OKINAWA SCHOOL IN PHYSICS: COHERENT QUANTUM DYNAMICS
Sep.28th-Oct. 9th @OIST

Interdisciplinary fields between quantum information science and physics

Keisuke Fujii
The Hakubi center for advanced research/
Graduate School of Science
Kyoto University

Outline of 3 Days

Lecture 1: foundations of quantum computation

-elementary gates and universal quantum computation

-quantum algorithms

-quantum stabilizer formalism (graph state, quantum error correction)

Lecture 2: 1D quantum system

-what is quantum phase

-how it is useful for QIP

Lecture 3: 2D quantum system

-topologically ordered system

-how it is related to quantum error correction codes

-how topologically protected quantum computation works

Application of stabilizer formalism: measurement-based QC

Stabilizer formalizm

Stabilizer group: $\mathcal{S}_n \subset \mathcal{P}_n$ Hermitian Abelian subgroup

stabilizer group is specified by the set of generators $\langle \{S_i\} \rangle$.

Stabilizer code states:

$$|\Psi\rangle = S_i|\Psi\rangle \quad \text{for all } S_i \in \mathcal{S}_n$$

Logical operators: commute with / independent of the stabilizer group

ex) $\mathcal{S} = \langle ZZI, IZZ \rangle, \;\; L_X = XXX, L_Z = IIZ$

→ $\{|000\rangle, |111\rangle\}$ $\quad IIZ|111\rangle = -|111\rangle, \;\; XXX|000\rangle = |111\rangle$

logical operators act nontrivially inside the code space

Graph (cluster) state

◆ Definition of a graph state

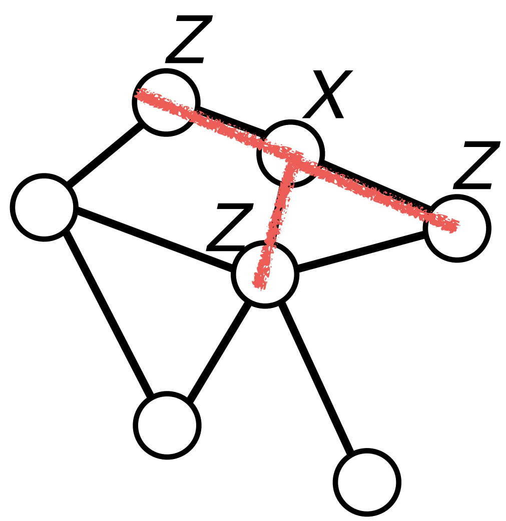
A stabilizer generator is defined for each vetex

$$K_i = X_i \prod_{j \sim i} Z_j$$

$$K_i |G\rangle = |G\rangle \text{ for all } i \in V$$

graph $G=(V,E)$

V: vertices, E:edges



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CZ gate

$$|G\rangle = \prod_{e \in E} \Lambda_e(Z) |+\rangle^{\otimes |V|}$$

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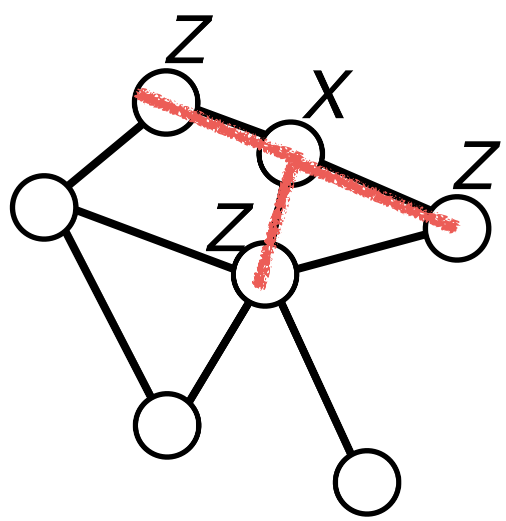

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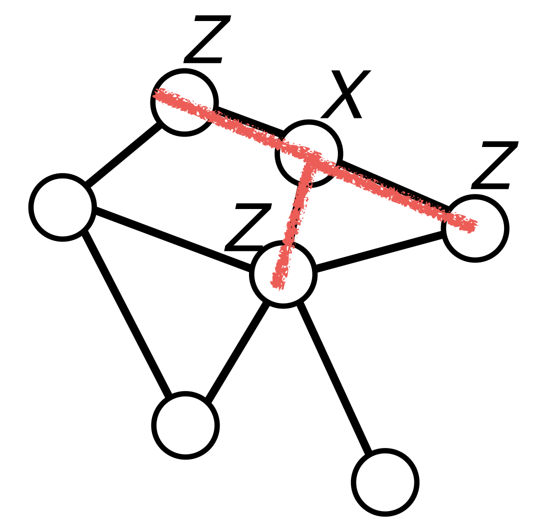
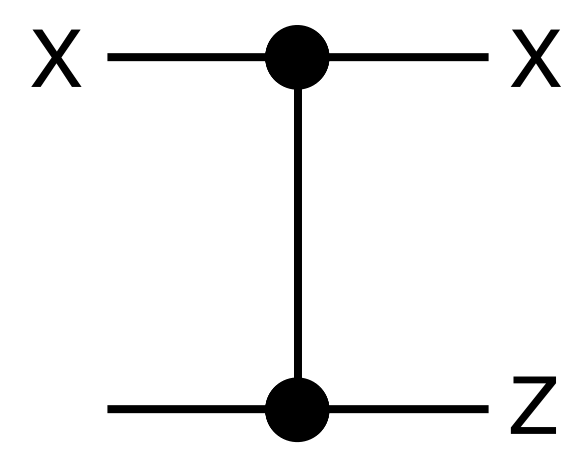


CZ gate

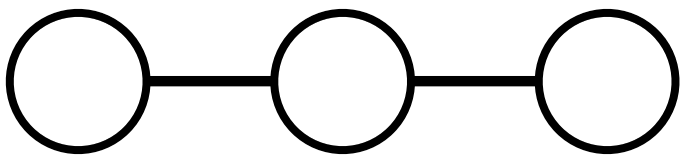
$$|G\rangle = \prod_{e \in E} \Lambda_e(Z)|+\rangle^{\otimes|V|}$$

$$\left(K_i = \left[\prod_{e \in E} \Lambda(Z)\right] X_i \left[\prod_{e \in E} \Lambda(Z)\right] \right)$$



1D graph (cluster) state

- 3-qubit 1D graph state



X — Z

Z — X — Z

Z — X

$$\frac{1}{\sqrt{2}}(|+\rangle|0\rangle|+\rangle + |-\rangle|1\rangle|-\rangle)$$

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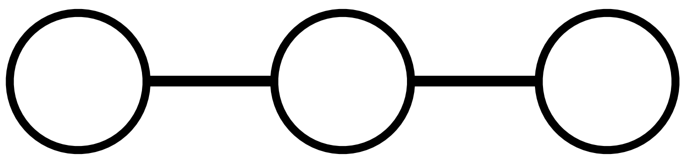
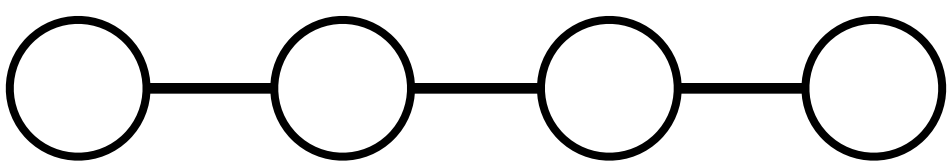


$$
\begin{array}{ccccc}
X & \text{——} & Z & & \\
Z & \text{——} & X & \text{——} & Z \\
 & & Z & \text{——} & X
\end{array}
$$

$$\frac{1}{\sqrt{2}}(|+\rangle|0\rangle|+\rangle + |-\rangle|1\rangle|-\rangle)$$

- 4-qubit 1D graph state



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X & \text{——} & Z & & & & \\
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 & & Z & \text{——} & X & \text{——} & Z \\
 & & & & Z & \text{——} & X
\end{array}
$$

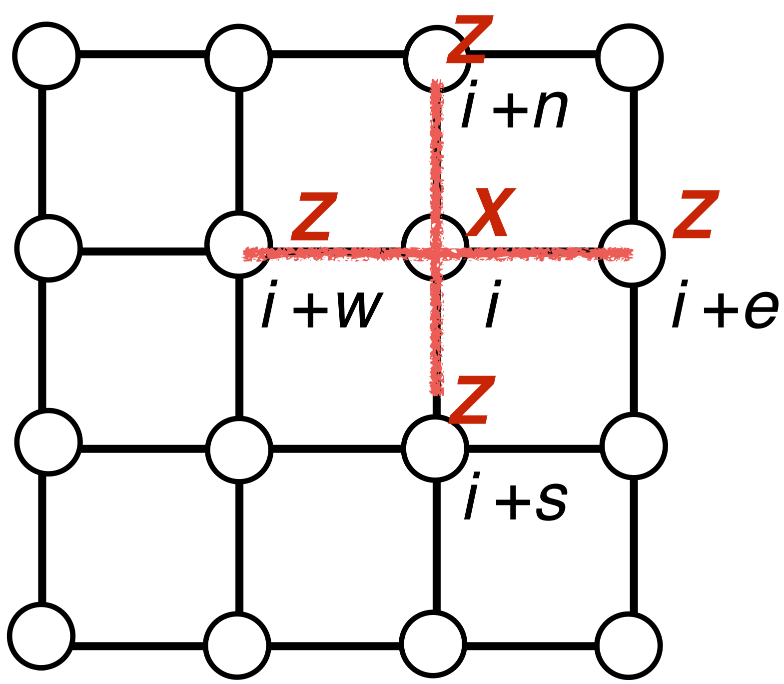
$$\frac{1}{2}(|+\rangle|0\rangle|0\rangle|+\rangle + |+\rangle|0\rangle|1\rangle|-\rangle$$
$$+ |-\rangle|1\rangle|0\rangle|+\rangle - |-\rangle|1\rangle|1\rangle|-\rangle)$$

MBQC

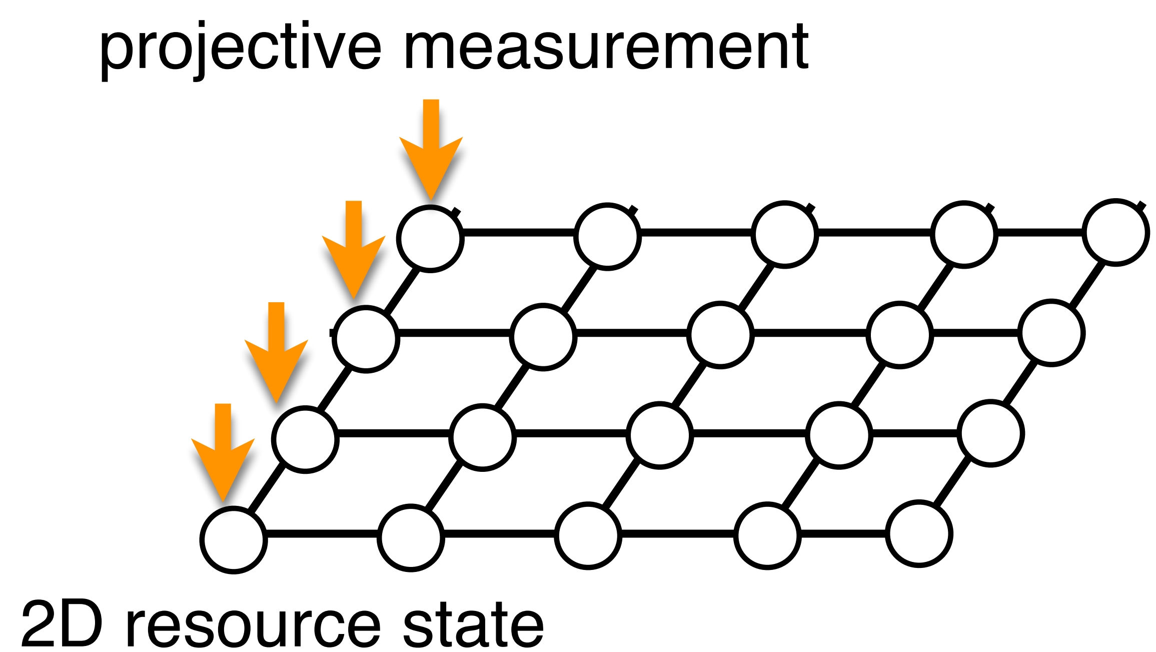
measurement-based quantum computation

Raussendorf-Briegel PRL **86** 910 (2001); Raussendorf-Browne-Briegel PRA **68** 022312 (2003).

- 2D cluster state



$$K_i = X_i Z_{i+n} Z_{i+e} Z_{i+s} Z_{i+w}$$



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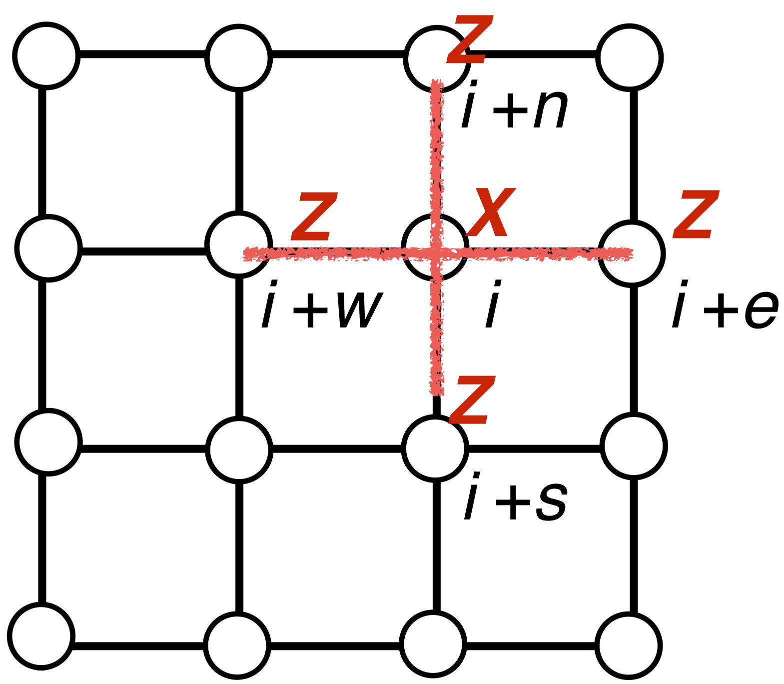

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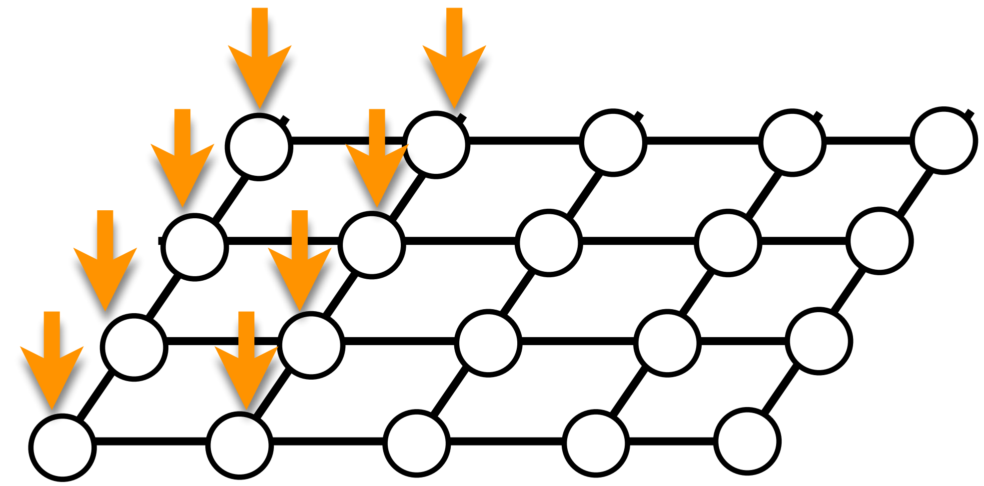
projective measurement



2D resource state

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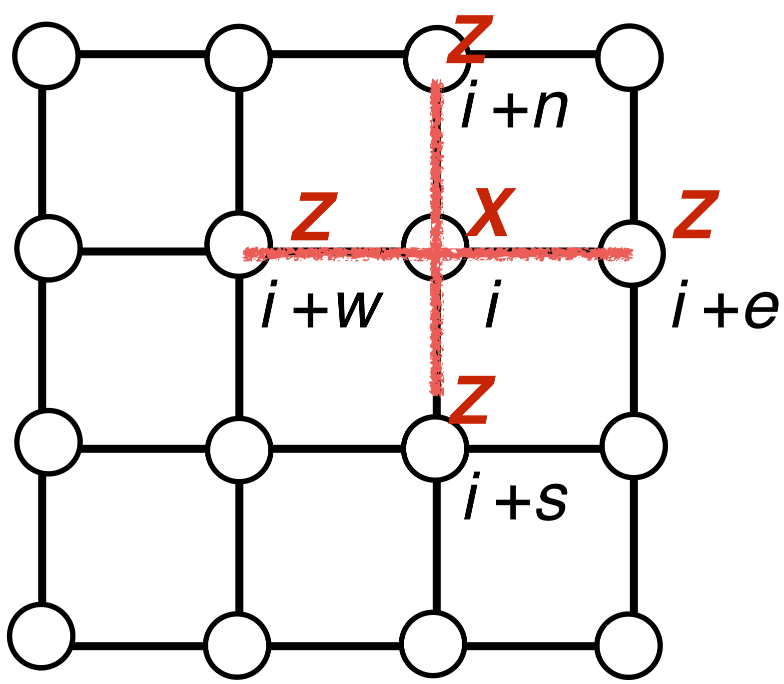

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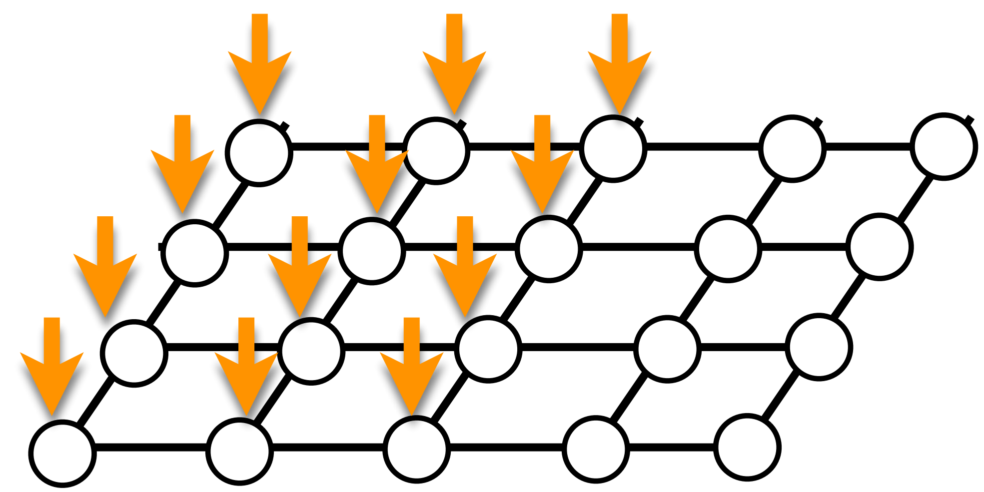

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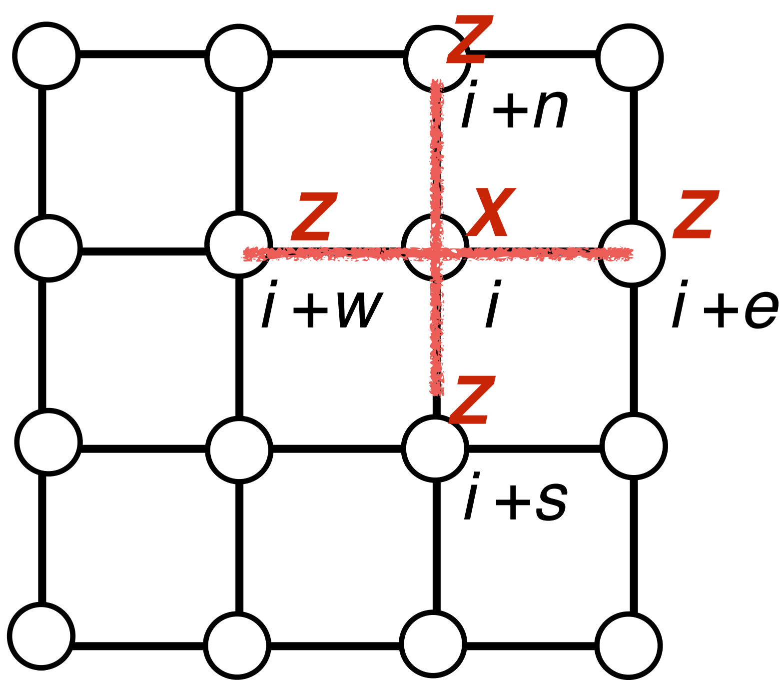

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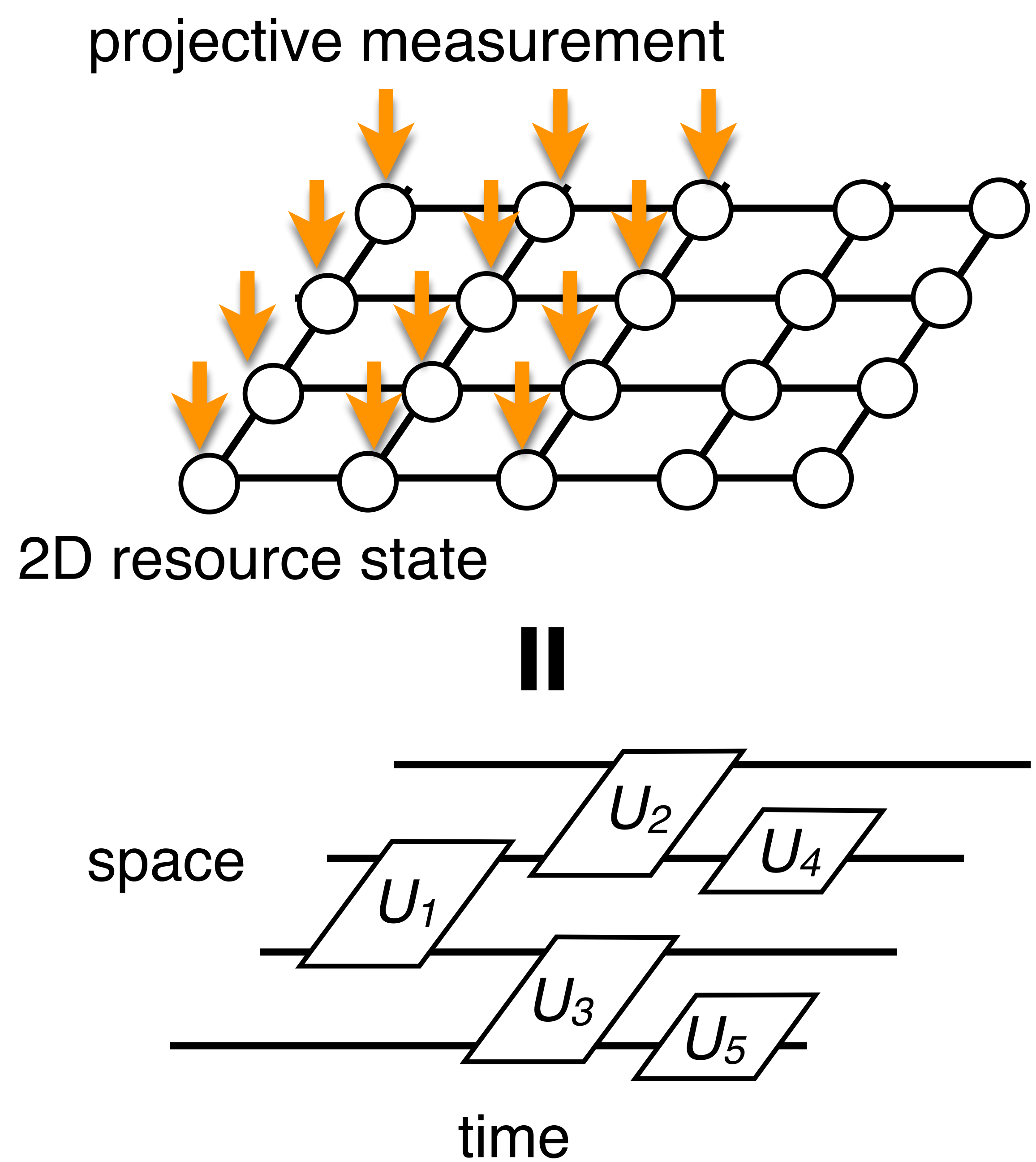

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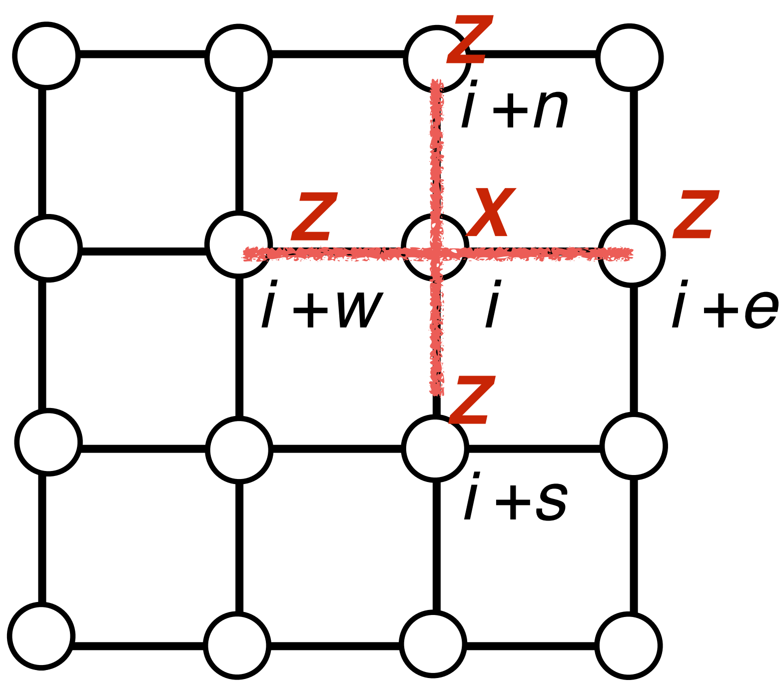

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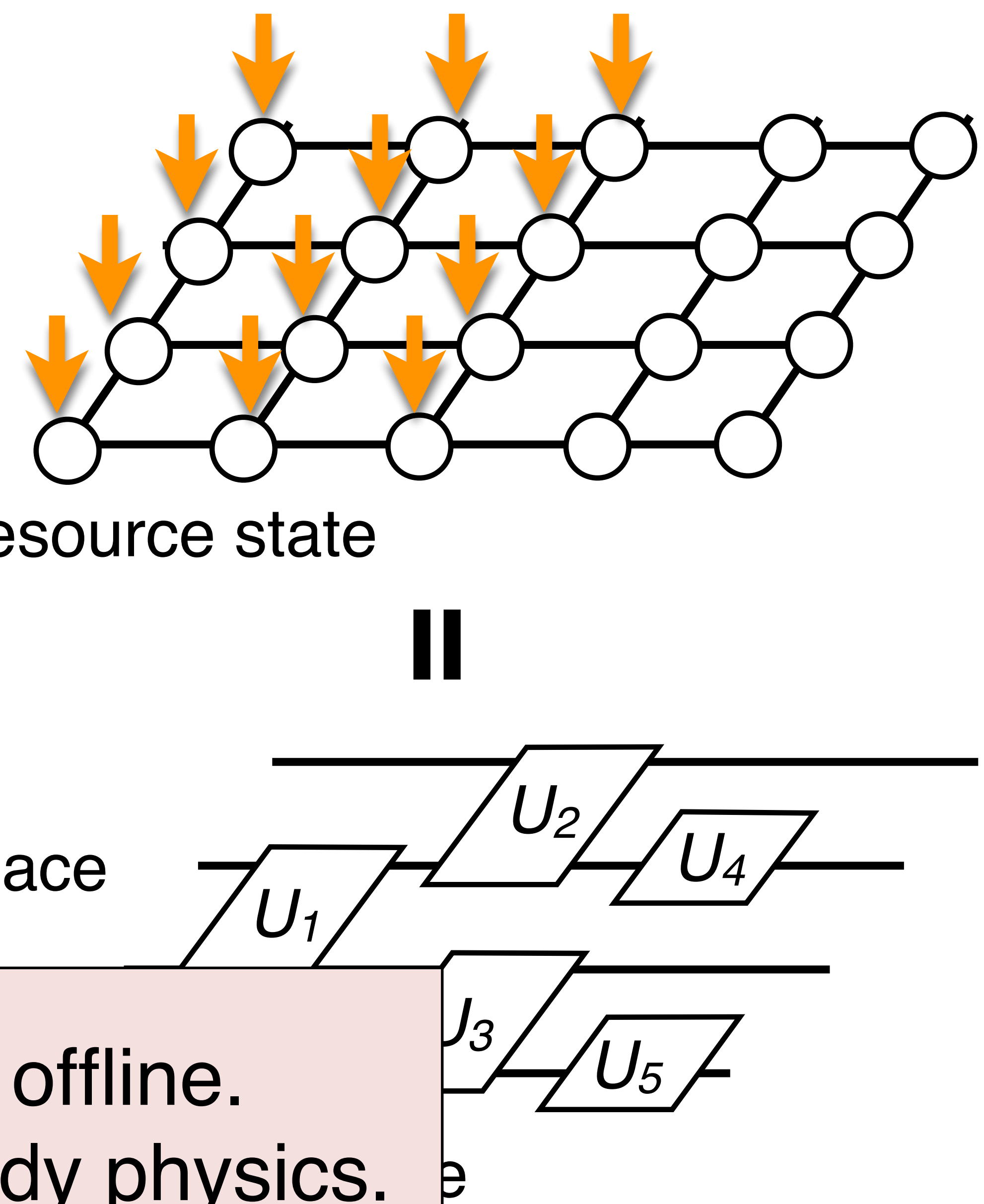

- Entangling operations are required only offline.
- Provide a connection between many-body physics.

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Today's topic

What is quantum phase?

How is it useful for QIP?

How is it realized in a physically natural 1D system?

Keywords: Majorana fermion, symmetry protected topological order, AKLT state

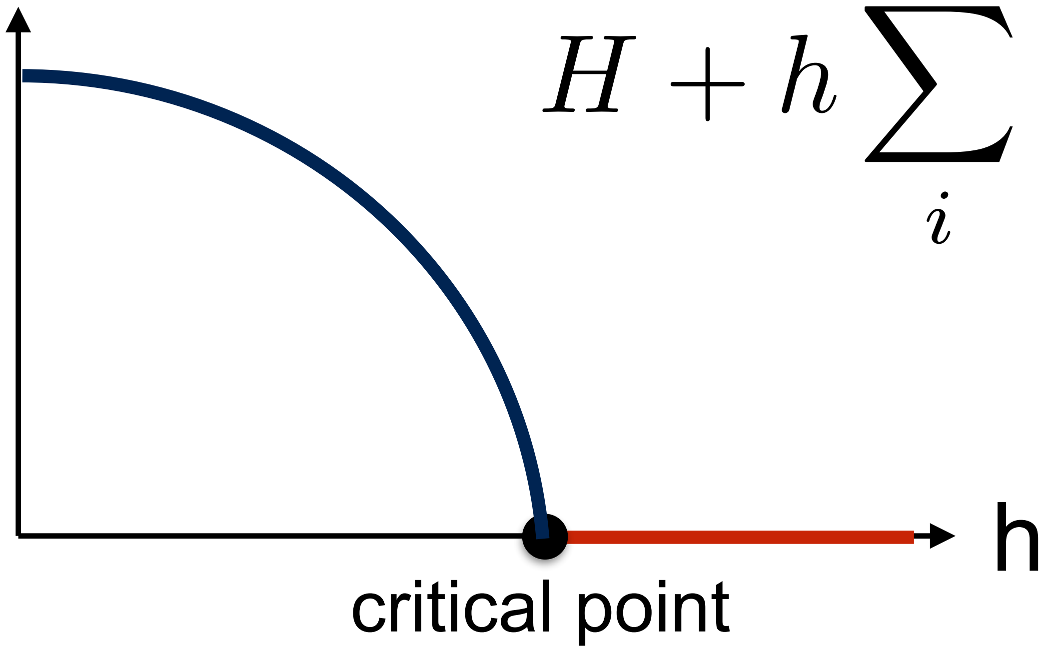
What is quantum phase?

- Quantum phase is a property of **ground state (g.s.)** of many-body systems.

$$H + h\sum_i Z_i$$

$$\langle O \rangle = \langle \Psi_{\mathrm{g.s.}} | O | \Psi_{\mathrm{g.s.}} \rangle$$

order parameter



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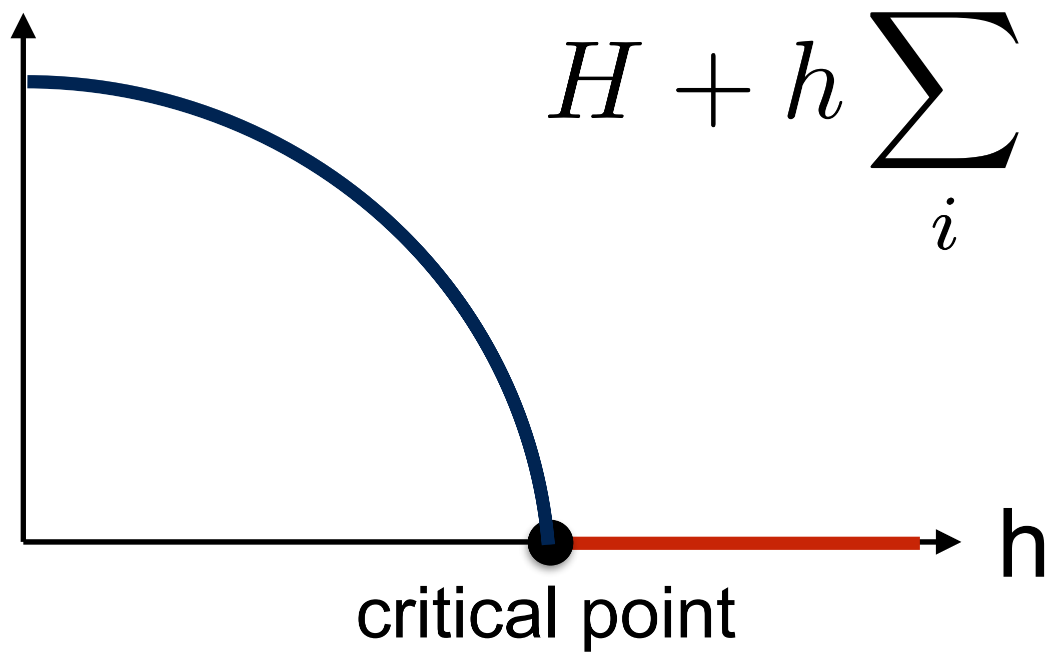

- The concept of “phase” is robust, and hence it would be useful for quantum information processing.

Degeneracy in g.s.

- The **degeneracy** of g.s. and its robustness against perturbation.

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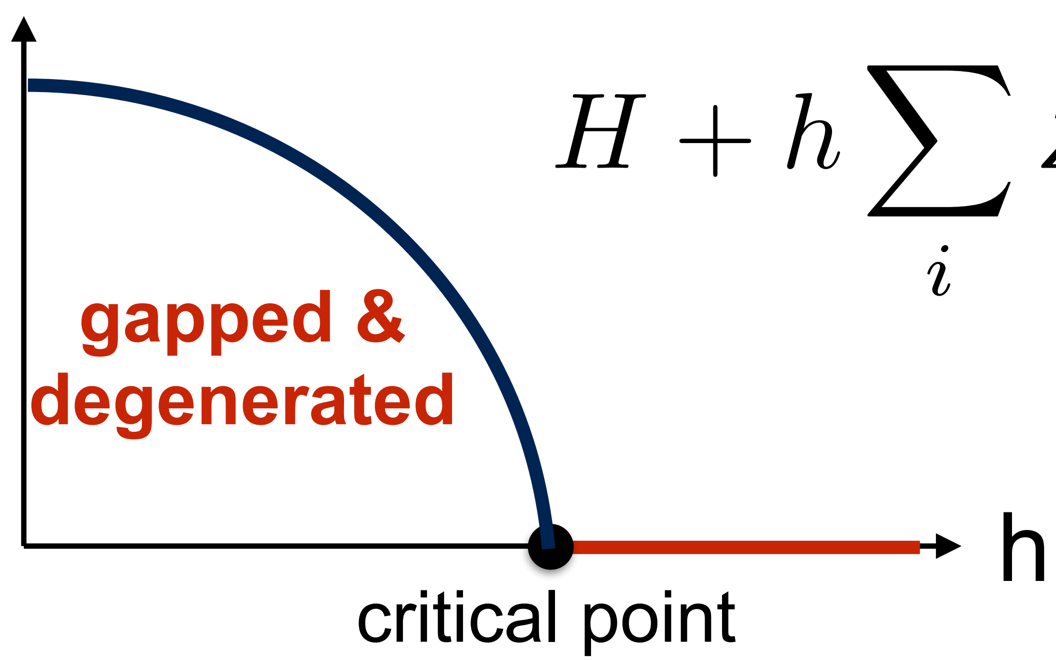

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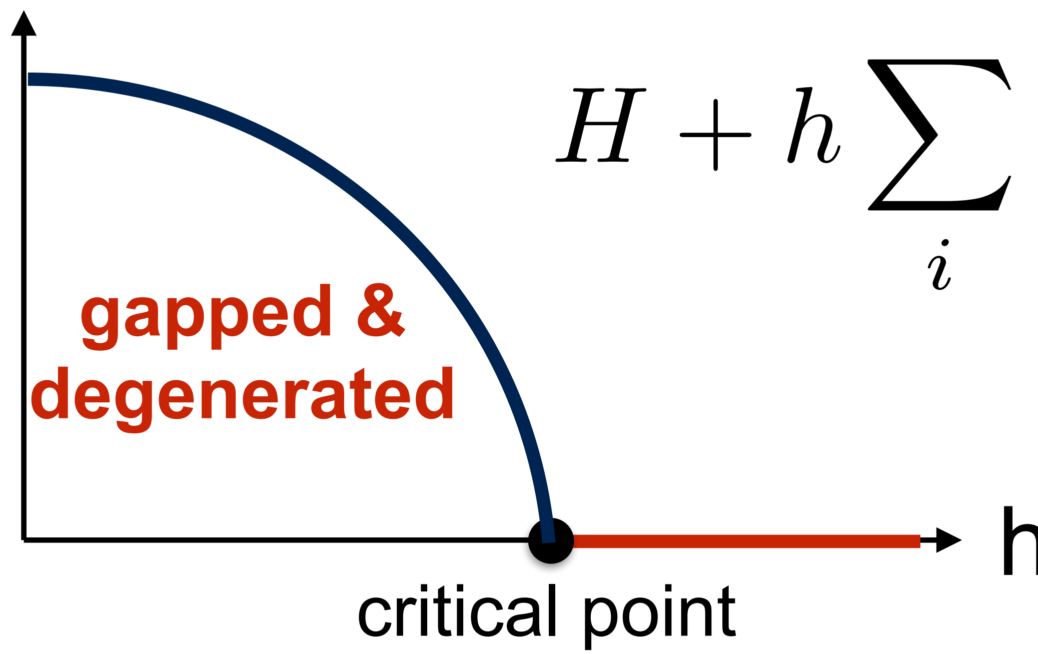

$$H + h \sum_i Z_i$$

- Quantum information can be encoded into the g.s. and computation can be done inside the g.s.

$\rightarrow$ **how robust? & how computation is done?**

1D Ising model

- Ising model with open boundary condition:

$$H_{\text{Ising}} = -\sum_{i=1}^{N-1} Z_i Z_{i+1}$$

(recall that $ZZ|00\rangle = |00\rangle, ZZ|11\rangle = |11\rangle$)

- The g.s. is degenerated: $\{|0...0\rangle, |1...1\rangle\}$

$$S = \prod_i X_i \qquad S H_{\text{Ising}} S^\dagger = H_{\text{Ising}}$$

global spin flip (Z2)

1D Ising model

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stabilizer generators

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code space

logical operator

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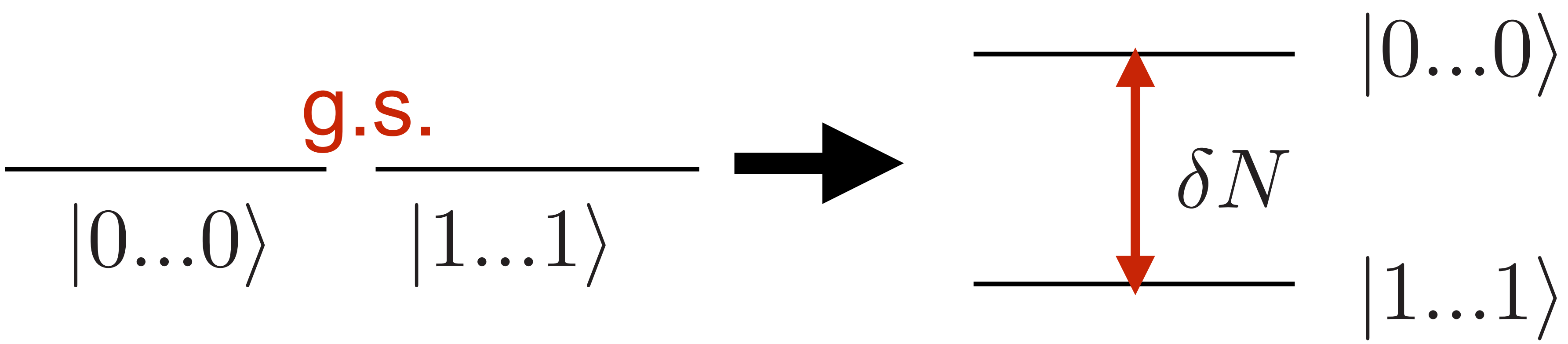

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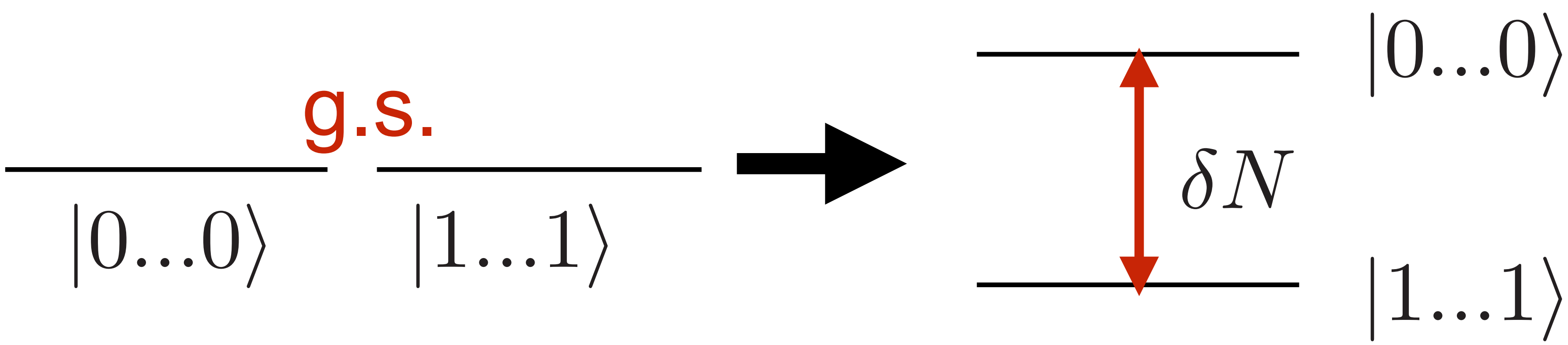

In the large *N* limit, the g.s.d. is lifted down, so is not protected against perturbations.

Is there a g.s.degeneracy which is robust against perturbations?

Yes.→ topologically ordered system

What is topological order

-a new kind of order in zero-temperature phase of matter.

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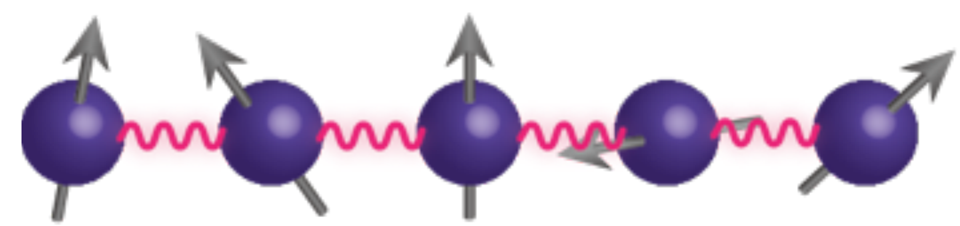
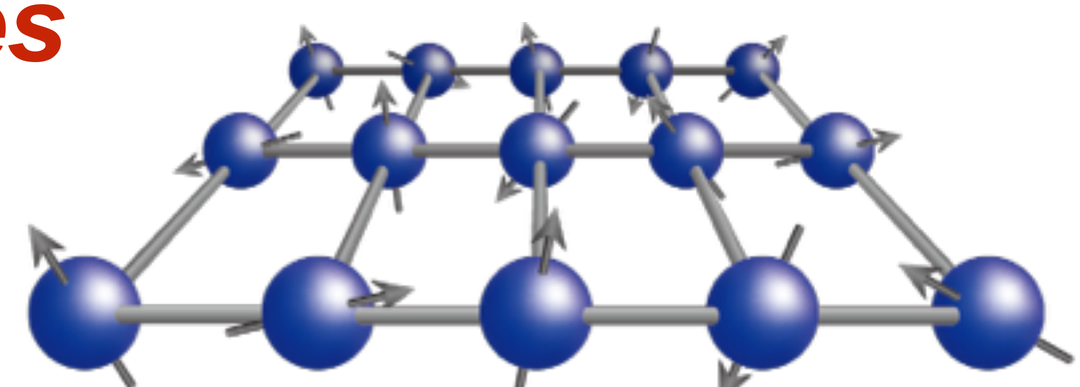
-related to quantum spin liquids, fractional quantum Hall effect, fault-tolerant quantum computation.

Outline of Lecture 2,3

Today:
symmetry protected topological order in 1D quantum many-body system



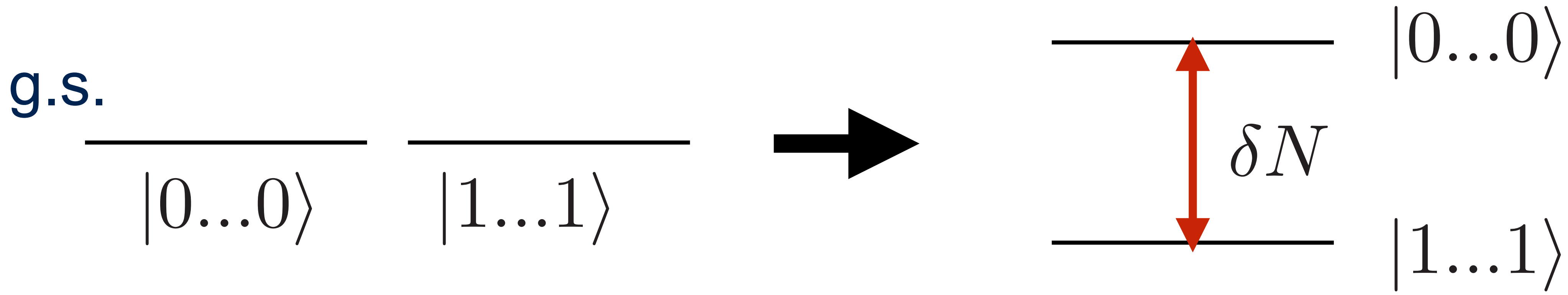
Tomorrow:
genuinely topologically ordered system in 2D quantum many-body system and quantum error correction codes



1D Ising model

$$H' = -\sum_{i=1}^{N-1} Z_i Z_{i+1} + \delta \sum_i Z_i$$

what if this kind of perturbation
is prohibited by …. symmetry



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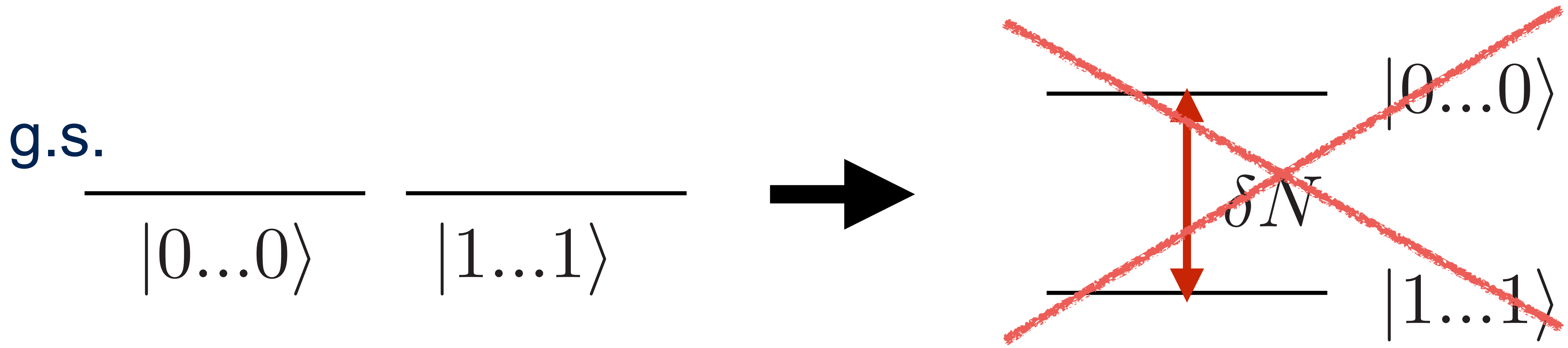

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1D Majorana fermion

Let us consider a mathematically equivalent but physically different system.

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Jordan-Wigner transformation
(spin $\rightleftarrows$ fermion)

$$\hat{a}_{2i-1} = X_1 ... X_{i-1} Z_i$$

$$\hat{a}_{2i} = X_1 ... X_{i-1} Y_i$$

$$\{\hat{a}_k, \hat{a}_{k'}\} = 2\delta_{k,k'} I$$

(Majorana fermion operator)

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(Majorana fermion operator)

2*N* spinless Majorana fermions: $H_{\mathrm{Maj}} = -\sum_{j=2}^{N-1} (-i)\hat{a}_{2j}\hat{a}_{2j+1}$

p-wave superconductor, topological insulator, semiconducting heterostructure
(see A. Kitaev and C. Laumann, arXiv:0904.2771 for review)

1D Majorana fermion

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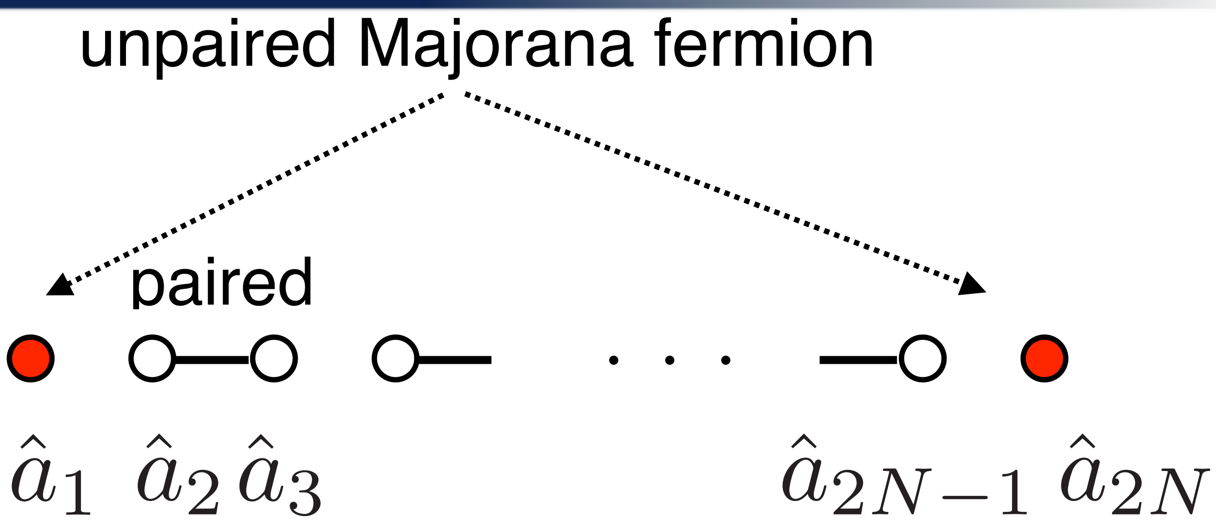

ground states: $(-i)\hat{a}_{2i}\hat{a}_{2i+1}|\Psi\rangle = |\Psi\rangle$ for all *i*.

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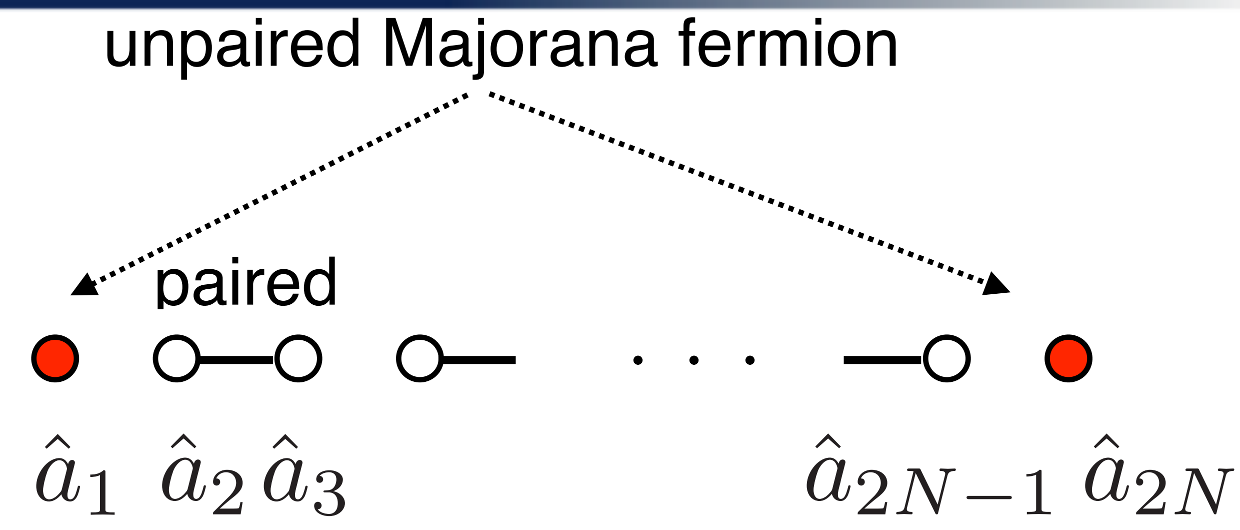

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unpaired Majorana fermions at the edges of the chain.

→ “zero-energy Majorana boundary mode” $\{|\bar{0}\rangle, |\bar{1}\rangle\}$

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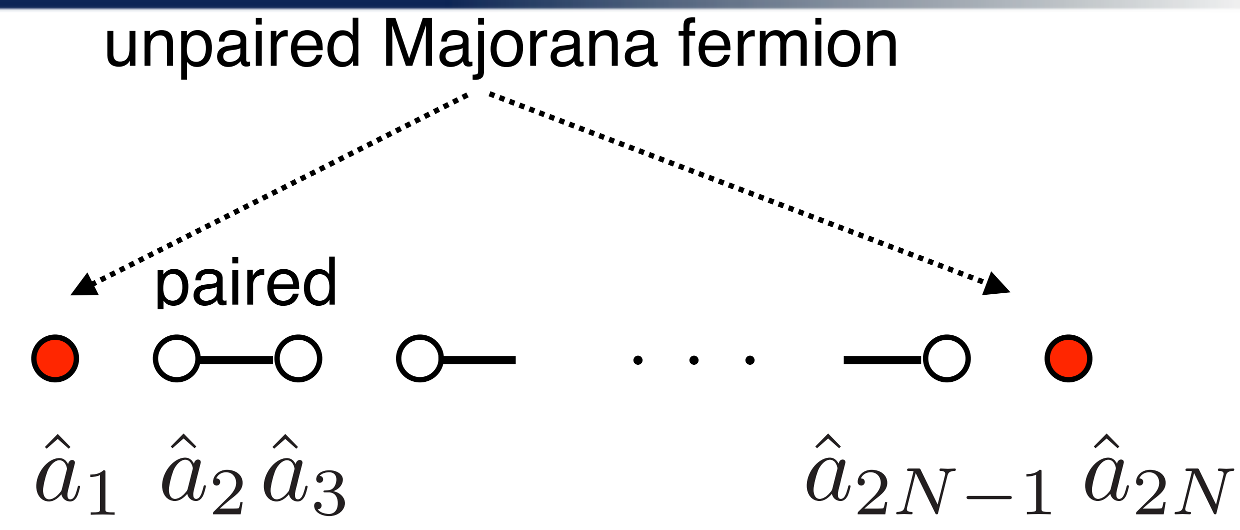

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$$(-i)\hat{a}_1\hat{a}_{2N}|\bar{0}\rangle = |\bar{1}\rangle, \quad (-i)\hat{a}_1\hat{a}_{2N}|\bar{1}\rangle = |\bar{0}\rangle, \quad \hat{a}_1|\bar{1}\rangle = -|\bar{1}\rangle$$

$\downarrow$ $Y_1 X_2 \ldots X_{2N-1} Y_{2N}$ (Z2 symmetry)

If unpaired Majonara fermions are well separated, this operator would not act.

$\downarrow$ Z_1

(act on the ground subspace nontrivially)

but fermion operators always appear as a pair!

1D Majorana fermion

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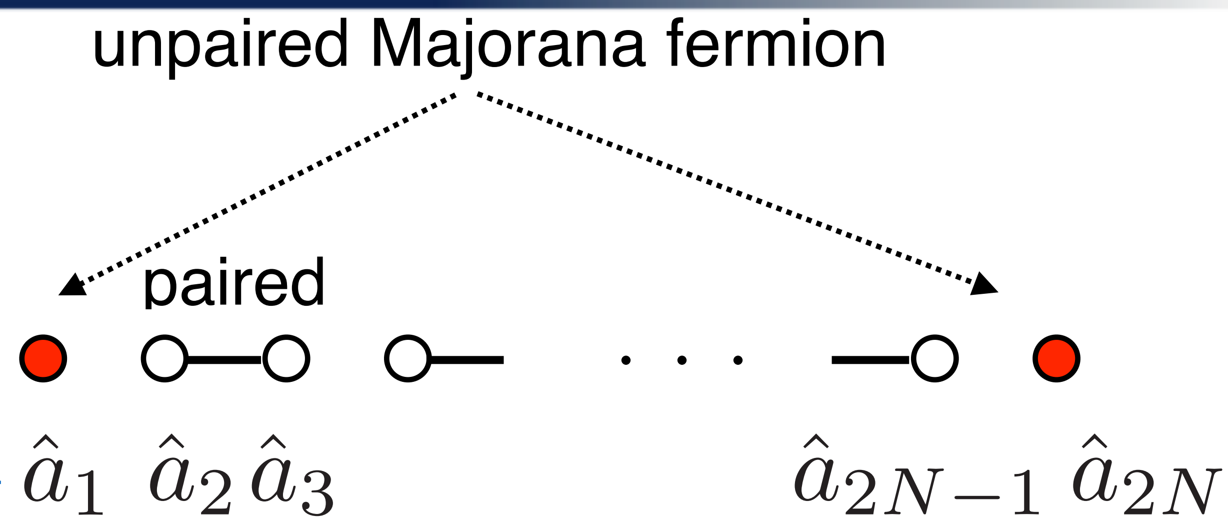
stabilizer generators



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logical operator (low weight) =prohibited

$$(-i)\hat{a}_1\hat{a}_{2N}|\bar{0}\rangle = |\bar{1}\rangle, \quad (-i)\hat{a}_1\hat{a}_{2N}|\bar{1}\rangle = |\bar{0}\rangle, \quad \hat{a}_1|\bar{1}\rangle = -|\bar{1}\rangle$$

logical operator(high weight) =symmetry

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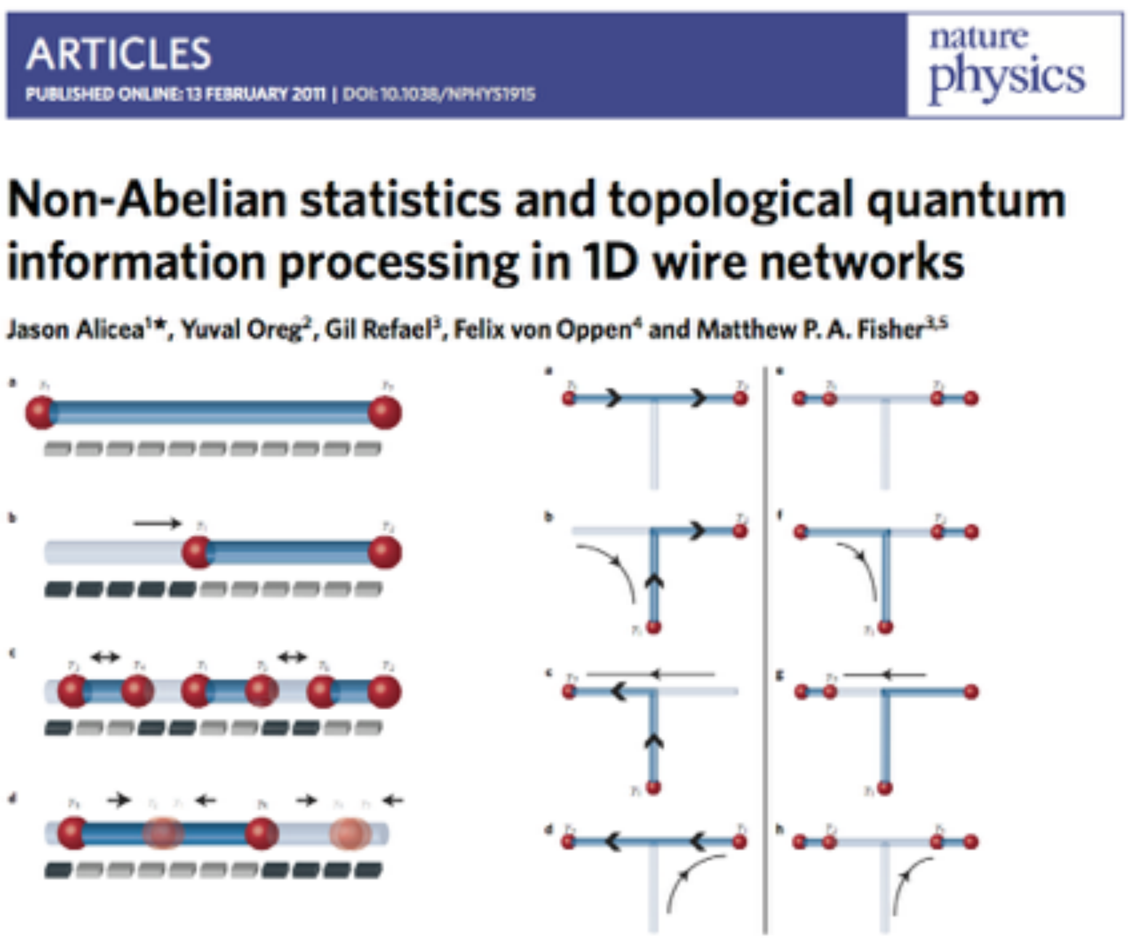
1D Majorana fermion

Unpaired Majorana fermions (g.s. degeneracy) is robust against any physical perturbation, which preserves the fermionic parity (symmetry).

Symmetry protected topological (SPT) order

Majorana fermion

1D p-wave superconductor; spin-orbit-coupled semiconducting wire on s-wave superconductor; topological insulator; cold atom in optical lattice



ARTICLES

nature physics

PUBLISHED ONLINE: 13 FEBRUARY 2011 | DOI: 10.1038/NPHYS1915

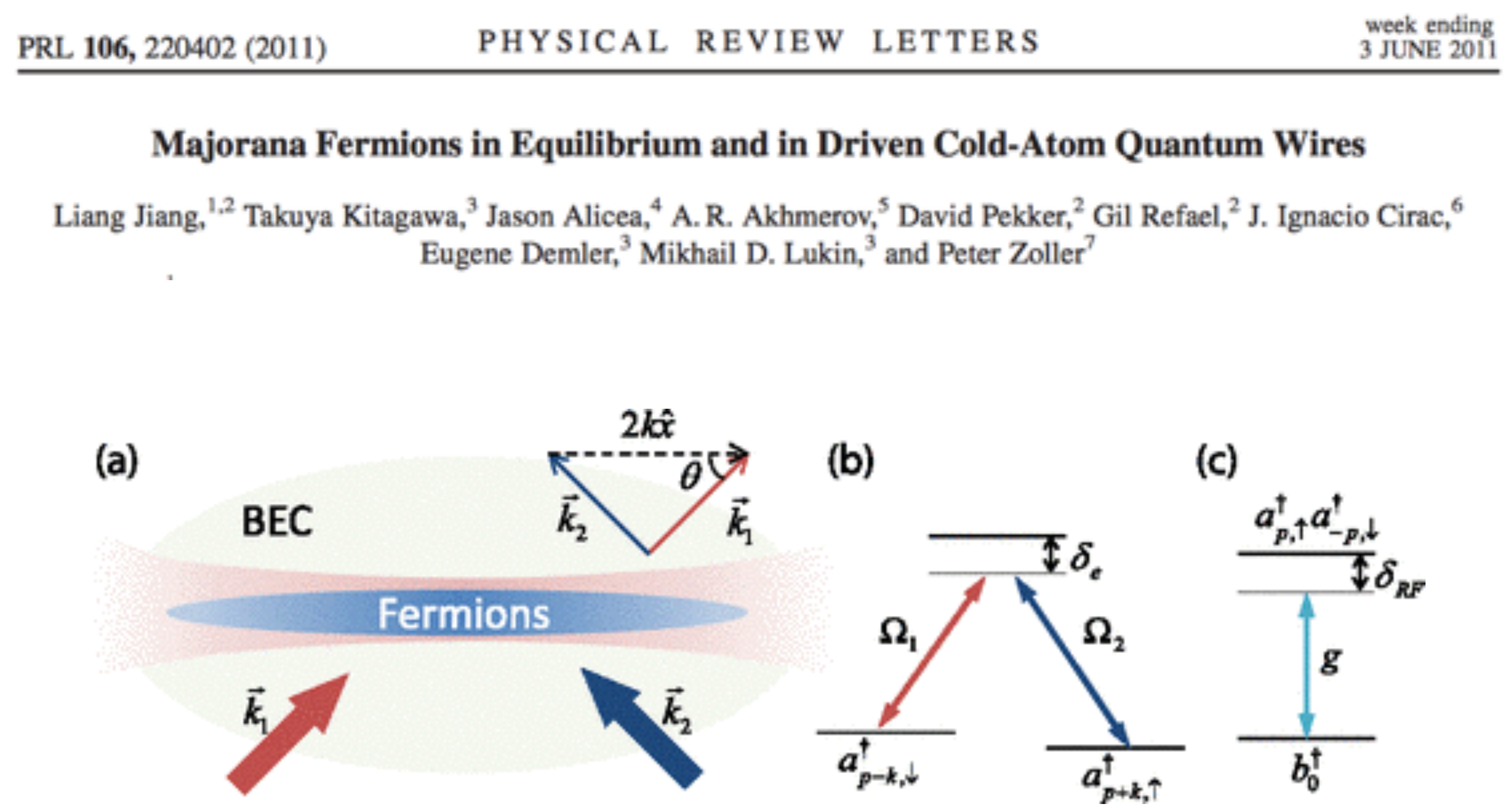
Non-Abelian statistics and topological quantum information processing in 1D wire networks

Jason Alicea[1*], Yuval Oreg[2], Gil Refael[3], Felix von Oppen[4] and Matthew P. A. Fisher[3,5]

pair creation of Majorana fermions

exchanging Majorana fermions via T-junction

[Alicea et al., Nature Physics 2011]



PRL 106, 220402 (2011) PHYSICAL REVIEW LETTERS week ending 3 JUNE 2011

Majorana Fermions in Equilibrium and in Driven Cold-Atom Quantum Wires

Liang Jiang,[1,2] Takuya Kitagawa,[3] Jason Alicea,[4] A. R. Akhmerov,[5] David Pekker,[2] Gil Refael,[2] J. Ignacio Cirac,[6] Eugene Demler,[3] Mikhail D. Lukin,[3] and Peter Zoller[7]

[Jiang et al., Phys. Rev. Lett 2011]

Majorana fermion

1D p-wave superconductor; spin-orbit-coupled semiconducting wire on s-wave superconductor; topological insulator; cold atom in optical lattice

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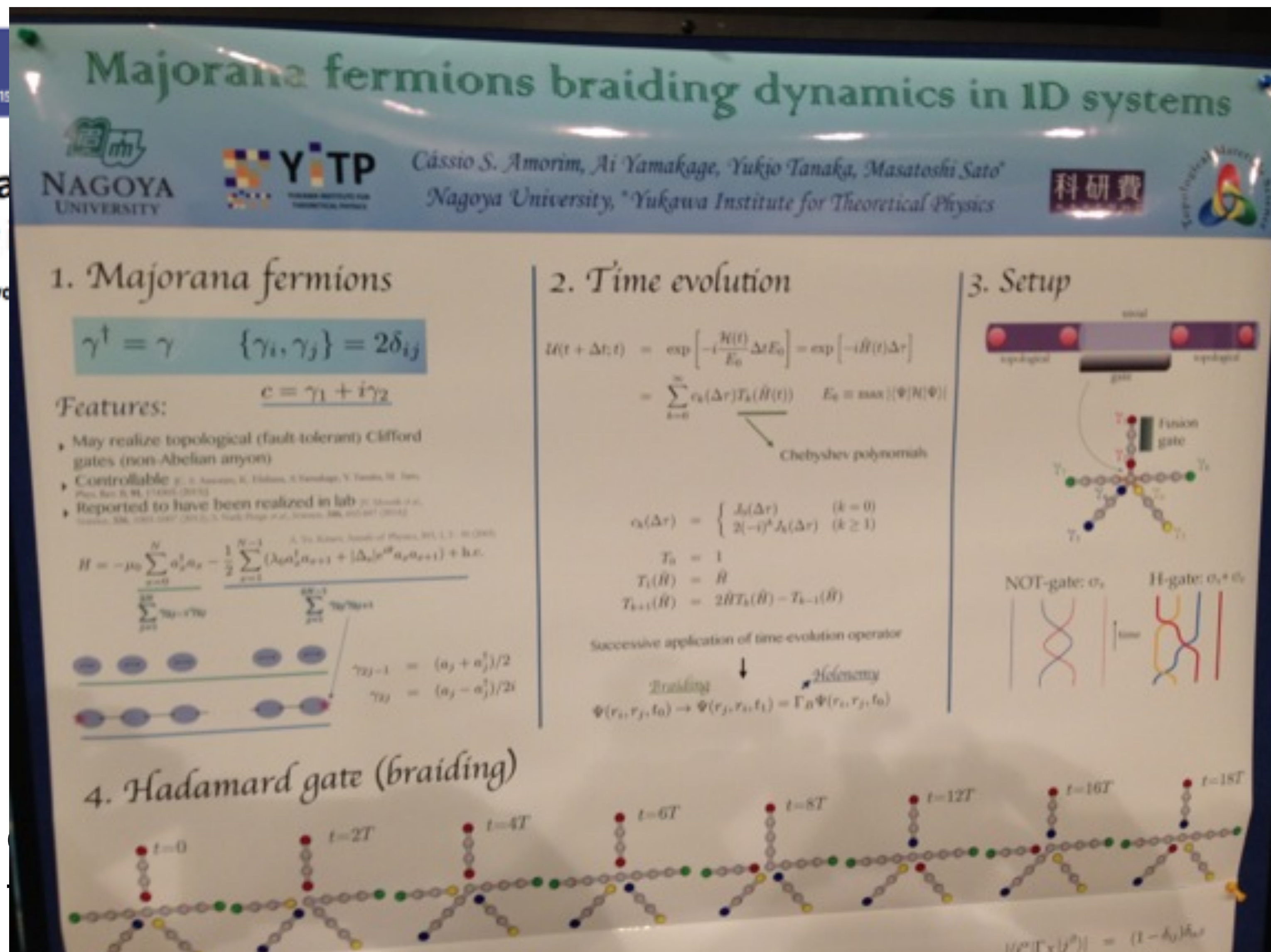
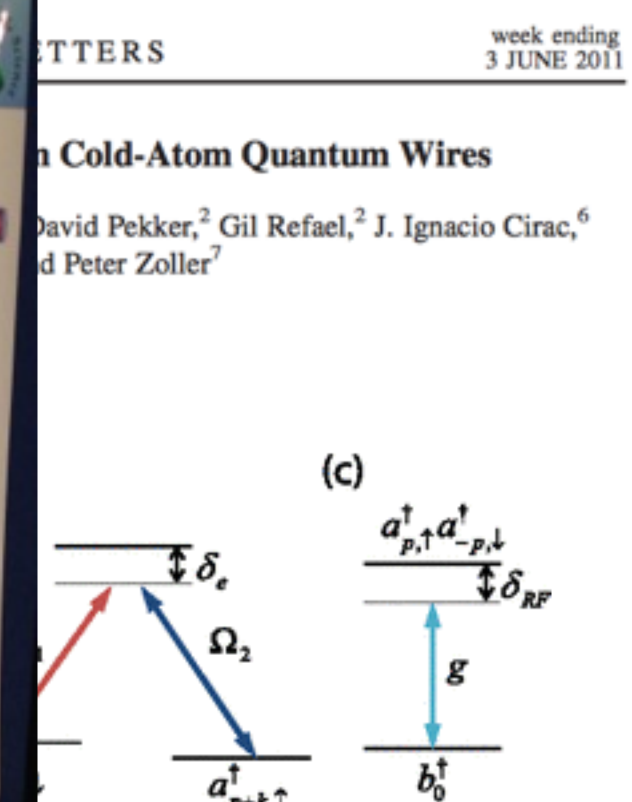


by Cassio Amorim



t 2011]

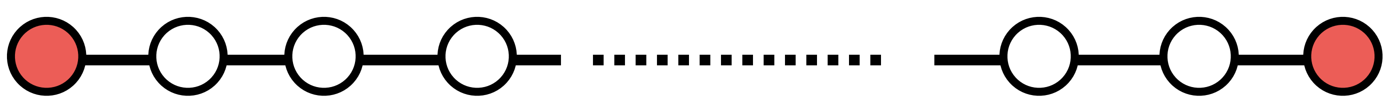
How SPT ordered states are useful for QIP?

1D Cluster Hamiltonian

$$H_{\mathrm{1dcluster}} = -\sum_{i=2}^{N-1} Z_{i-1} X_i Z_{i+1}$$

stabilizer generators

1D cluster state



$X_1 Z_2$

$Z_1 X_2 Z_3$ ....

$Z_{N-2} X_{N-1} Z_N$

$Z_{N-1} X_N$

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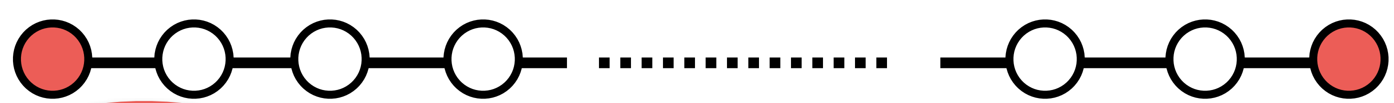

$X_1 Z_2$

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$Z_{N-1} X_N$

missing generators

→The g.s. has 4-fold degeneracy.

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1D Cluster Hamiltonian

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- Global spin flip operators on either odd or even qubits:

$$S_o := \prod_k X_{2k-1}, \quad S_e := \prod_k X_{2k}$$

- Z2×Z2 symmetry:

$$S_{\text{o}} H_{\text{1dcluster}} S_{\text{o}}^\dagger = H_{\text{1dcluster}}, \quad S_{\text{e}} H_{\text{1dcluster}} S_{\text{e}}^\dagger = H_{\text{1dcluster}}$$

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Any local perturbation commuting with the symmetry operators cannot lift down the g.s. degeneracy.

$$X_k, \quad Z_k Z_{k+2}$$

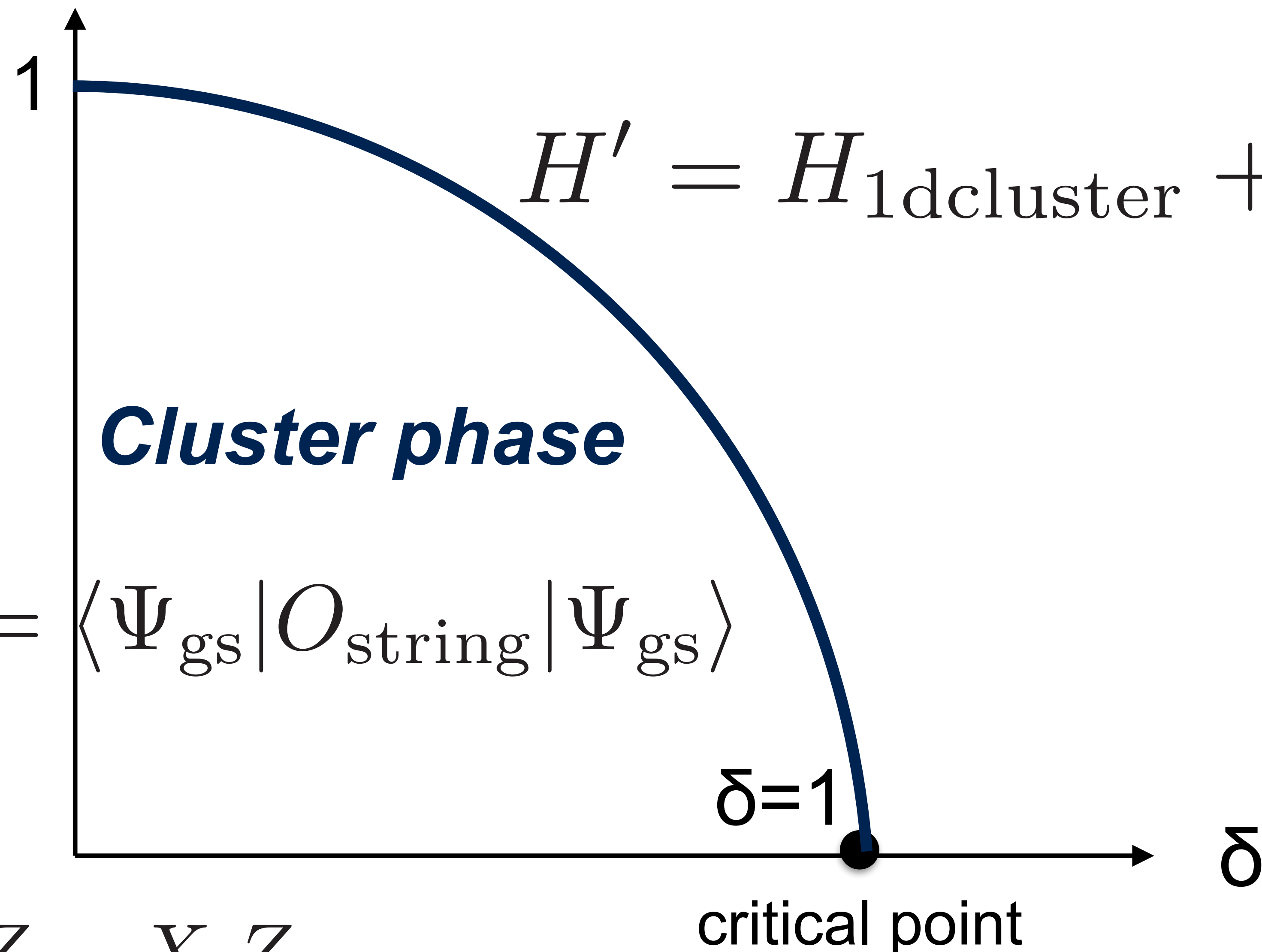
[J. Pachos and M. Plenio, PRL 2004; W. Son et al., EPL 2011]

Cluster Phase

$$H' = H_{\mathrm{1dcluster}} + \delta \sum_i X_i$$

String order parameter

$$\langle O_{\mathrm{string}} \rangle = \langle \Psi_{\mathrm{gs}} | O_{\mathrm{string}} | \Psi_{\mathrm{gs}} \rangle$$

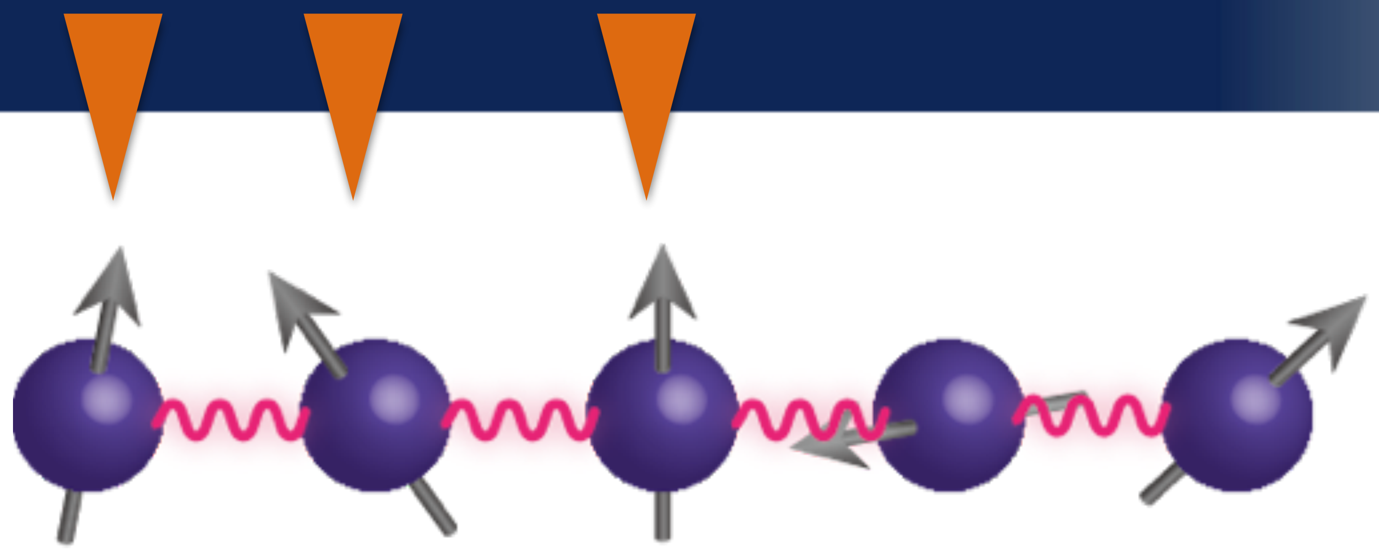


$$O_{\mathrm{string}} = \prod_i Z_{i-1} X_i Z_{i+1}$$

$$= Z_1 Y_2 X_3 \ldots X_{N-2} Y_{N-1} Z_N$$

[J. Pachos and M. Plenio, PRL 2004]

How computation is done in the g.s.?



Matrix product states

Fannes-Nachtergaele-Werner, Comm. Math. '92

- Description of quantum states are difficult in general.

$$|\Psi\rangle = \sum_{i_1,\ldots,i_N} C_{i_1,\ldots,i_N} |i_1\rangle|i_2\rangle\cdots|i_N\rangle$$

exponentially many coefficients!

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what if the coefficients have a nice structure?

$$|\Psi\rangle = \sum_{i_1,\ldots,i_N} \boxed{\langle R|A[i_n]\cdots A[i_2]A[i_1]|L\rangle} \times |i_1\rangle|i_2\rangle \cdots |i_N\rangle$$

the coefficient are denoted by matrix products

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virtual degree (correlation space)

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Examples of MPS

- GHZ state: $(|00...0\rangle + |11...1\rangle)/\sqrt{2}$

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$$\left[\prod_{i=1}^{N-1} \Lambda_{i,i+1}(Z)\right] |+\rangle^{\otimes N}$$

$$\Lambda(Z)|11\rangle = -|11\rangle$$

CZ put -1 for neighboring 11

all spin configuration

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pick up -1

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MBQC on 1D cluster model

$$|\Psi_{\text{clus}}\rangle = \sum_{i_1,\ldots,i_N} \langle R|A[i_N]\cdots A[i_2]A[i_1]|L\rangle|i_1 i_2 \ldots i_N\rangle$$

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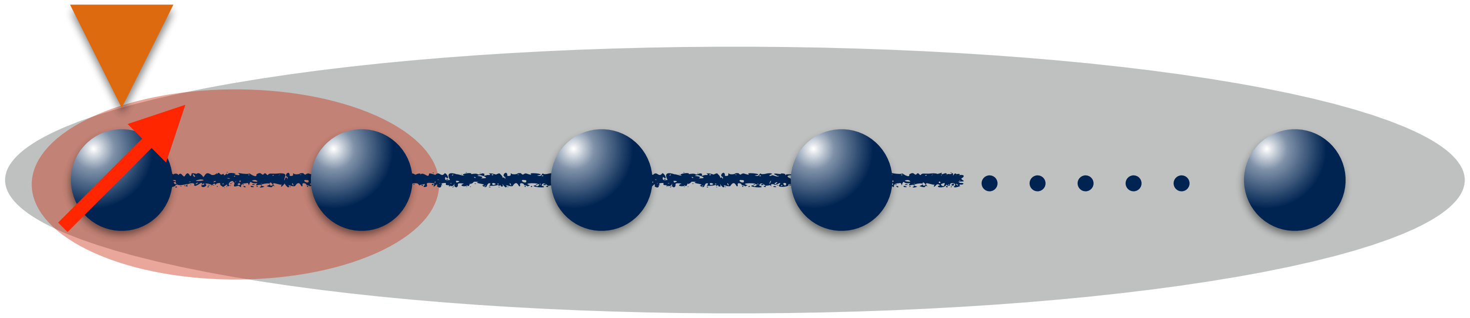
edge mode (degeneracy)

Projection on the 1st qubit in the basis $\left\{ \frac{1}{\sqrt{2}}(|0\rangle \pm e^{-i\phi}|1\rangle) \right\}$



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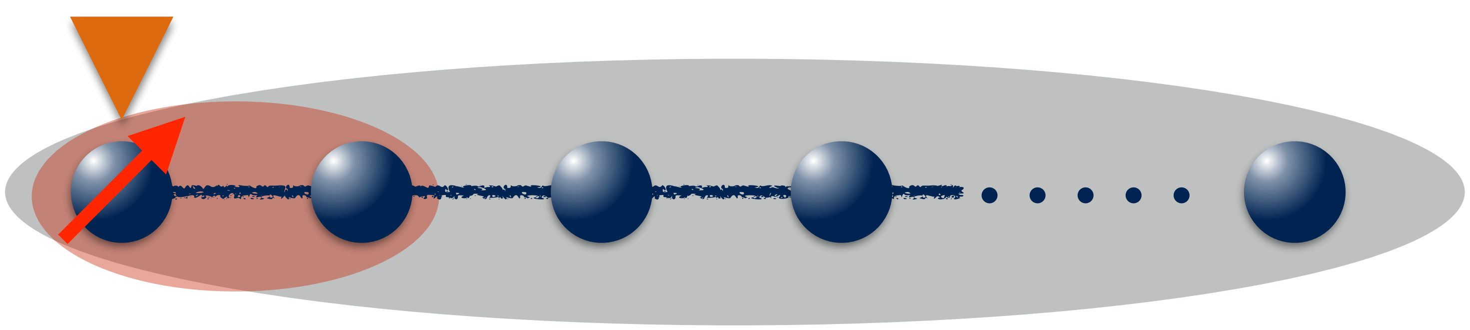


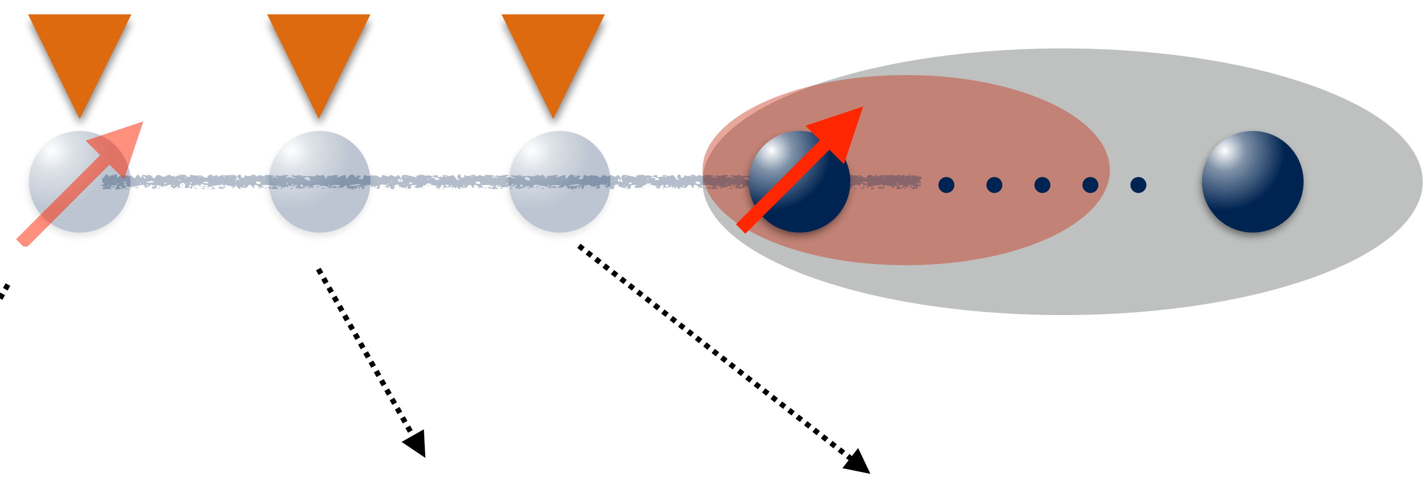
$$\frac{1}{\sqrt{2}}(\langle 0| + e^{i\phi}\langle 1|)|\Psi_{\mathrm{clus}}\rangle \qquad = He^{-i\phi Z/2}$$

$$= \sum_{i_2,\ldots,i_N} \langle R|A[i_N]\cdots A[i_2]\underline{(A[0] + e^{i\phi}A[1])/\sqrt{2}}|L\rangle|i_2\ldots i_N\rangle$$

$$= \sum_{i_2,\ldots,i_N} \langle R|A[i_N]\cdots A[i_2]\underline{|L'\rangle}|i_2\ldots i_N\rangle$$

$$\boxed{|L'\rangle := He^{-i\phi Z/2}|L\rangle}$$

MBQC on 1D cluster model

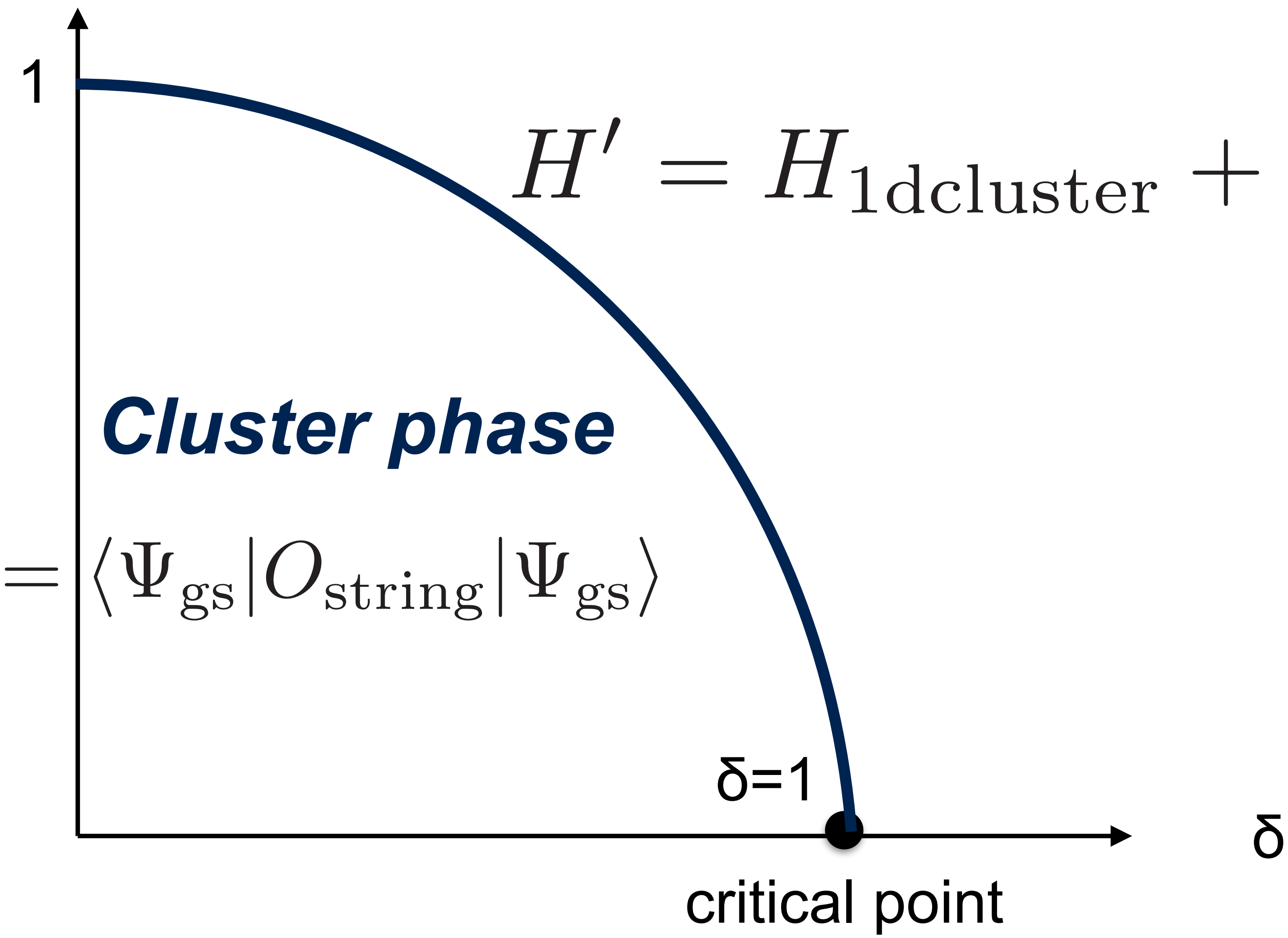


$$|L\rangle \;—\; \boxed{He^{-i\alpha Z/2}} \;—\; \boxed{He^{-i\beta Z/2}} \;—\; \boxed{He^{-i\gamma Z/2}} \;—$$

- An arbitrary SU(2) rotation can be implemented.
- Byproduct operator due to randomness of the measurement outcomes can be canceled by adoptive measurements.

MBQC on 1D cluster model

$$H' = H_{\mathrm{1dcluster}} + \delta \sum_i X_i$$



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[Pachos-Plenio, PRL '04; Doherty-Barrett PRL '09; Fujii et al., PRL '13]

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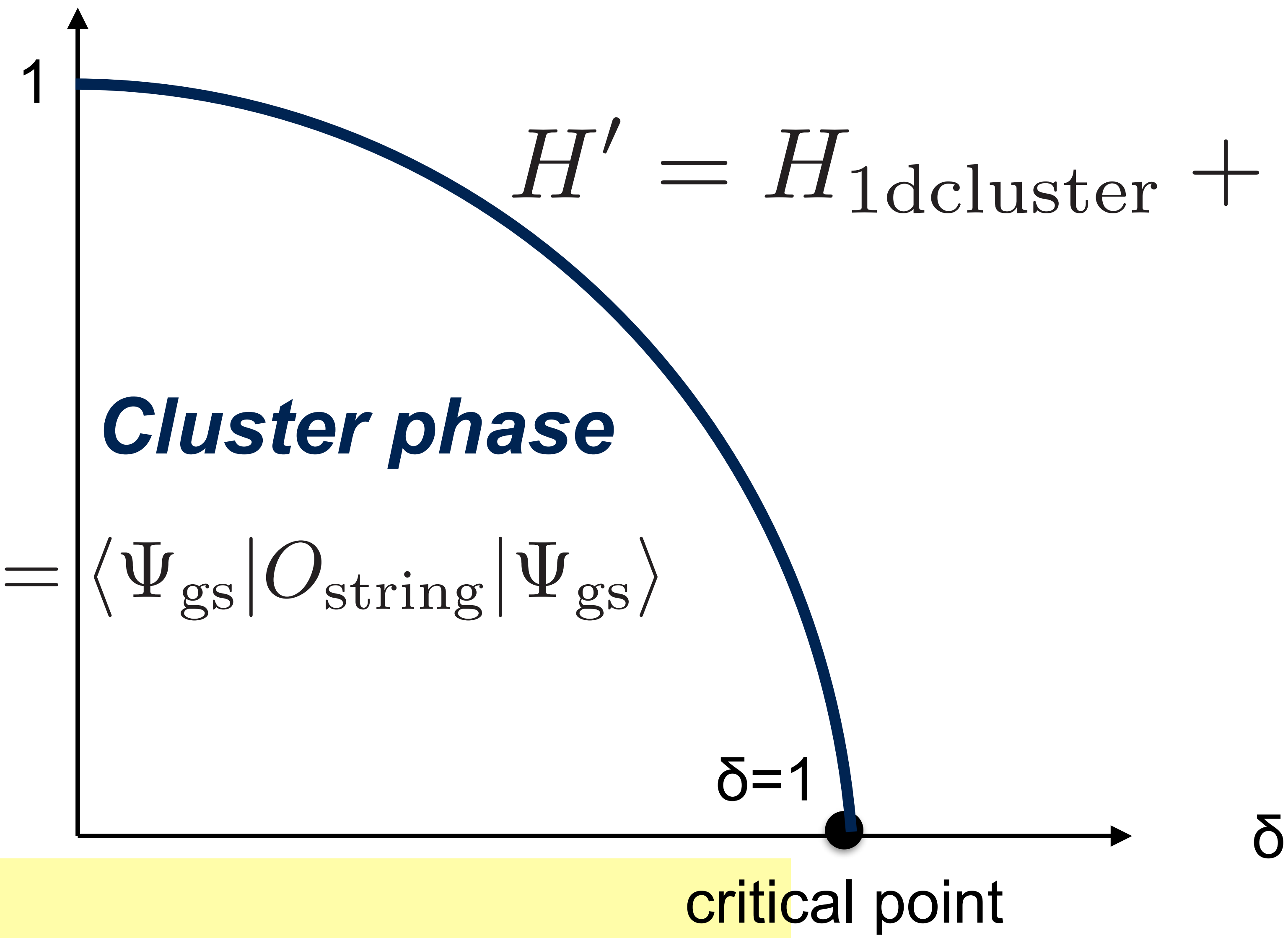


gate fidelity

$$F = \frac{(1 + \langle O^{\rm o}_{\rm string}\rangle)(1 + \langle O^{\rm e}_{\rm string}\rangle)}{4}$$

$$O^{\rm o,e}_{\rm string} = \prod_{\rm odd,even} Z_{i-1} X_i Z_{i+1}$$

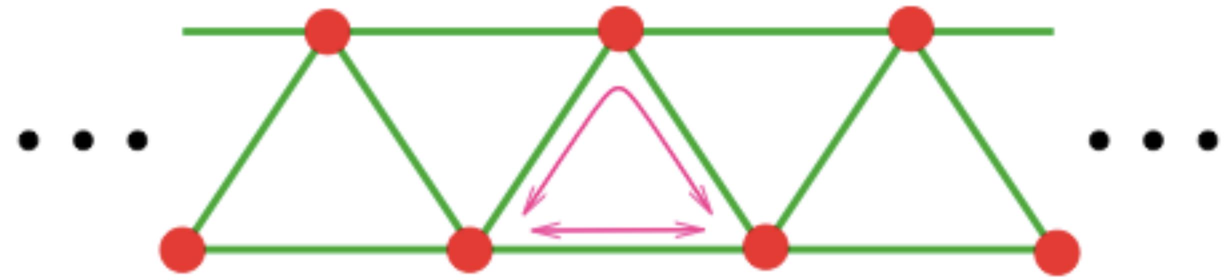
[Pachos-Plenio, PRL '04; Doherty-Barrett PRL '09; Fujii et al., PRL '13]

MBQC on 1D cluster model

VOLUME 93, NUMBER 5 PHYSICAL REVIEW LETTERS week ending 30 JULY 2004

Three-Spin Interactions in Optical Lattices and Criticality in Cluster Hamiltonians

Jiannis K. Pachos[1] and Martin B. Plenio[2]



tunneling interaction $$V = -\sum_{i\sigma}(J_i^\sigma a_{i\sigma}^\dagger a_{i+1\sigma} + \text{H.c.}),$$

3rd order perturbation $$H = -\sum_\gamma \frac{V_{\alpha\gamma}V_{\gamma\beta}}{E_\gamma} + \sum_{\gamma\delta} \frac{V_{\alpha\gamma}V_{\gamma\delta}V_{\delta\beta}}{E_\gamma E_\delta}.$$

$$\longrightarrow H_{\text{1dcluster}} = -\sum_{i=2}^{N-1} Z_{i-1}X_iZ_{i+1}$$

Can we obtain the resource from natural (two-body) Hamiltonian?

Can we obtain the resource from natural (two-body) Hamiltonian?

→spin-1 AKLT state

Haldane's conjecture

- Haldane's conjecture on 1D Heisenberg anti-ferro model:

$$H_{\mathrm{Heisenberg}} = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} \qquad \vec{S}_i = (S_i^x, S_i^y, S_i^z)$$

Odd half integer spins, 1/2, 3/2,…
→ massless(gapless), critical

Integer spins, 1, 2,…
→ massive (gapped), exponentially decaying correlation

AKLT-model

AKLT, PRL '87

- Spin-1 1D AKLT (Affleck-Kennedy-Lieb-Tasaki) Hamiltonian

$$H_{\mathrm{AKLT}} = \sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} + 1/3 (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right]$$

The g.s. is gapped, exhibits exponentially decaying correlation, **and is exactly given as an MPS!**

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$$i_k = 0, 1, 2$$

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2×2 matrices

$$A[0] = \sqrt{\frac{2}{3}}|0\rangle\langle 1|, \;\; A[1] = -\sqrt{\frac{1}{3}}Z, \;\; A[2] = -\sqrt{\frac{2}{3}}|1\rangle\langle 0|$$

AKLT-state

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$$\langle 00| \rightarrow \langle 0|$$

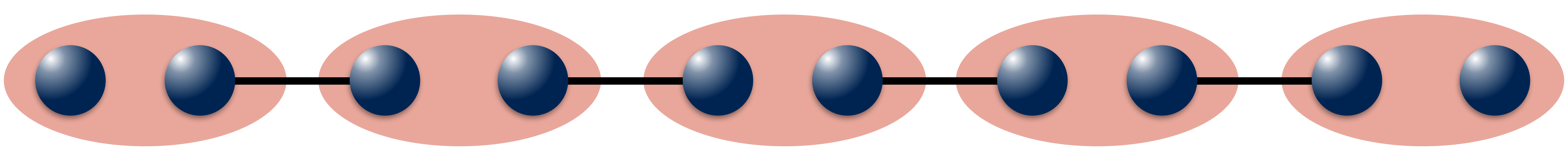
$$\frac{\langle 01| + \langle 10|}{\sqrt{2}} \rightarrow \langle 1|$$

projection
to spin-1 triplet
subspace

$$\langle 11| \rightarrow \langle 2|$$

edge state

edge state



spin-1/2
singlet

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

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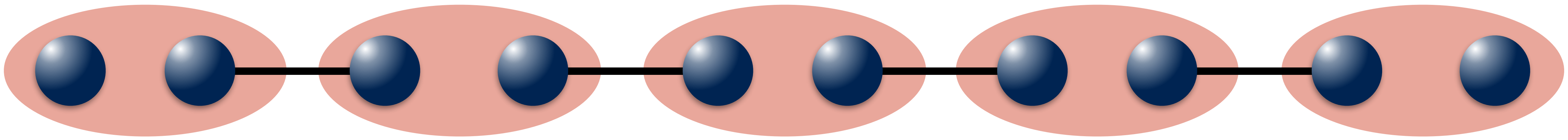

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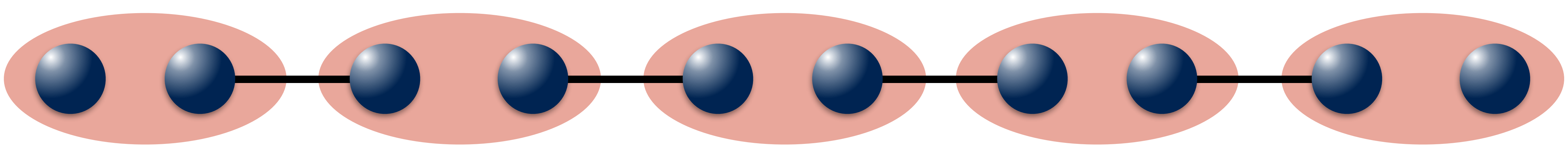
basis change = local unitary

$$\downarrow$$

$$B[\bar{0}] = \sqrt{\frac{1}{3}}X, \;\; B[\bar{1}] = \sqrt{\frac{1}{3}}Z, \;\; B[\bar{2}] = \sqrt{\frac{1}{3}}Y$$



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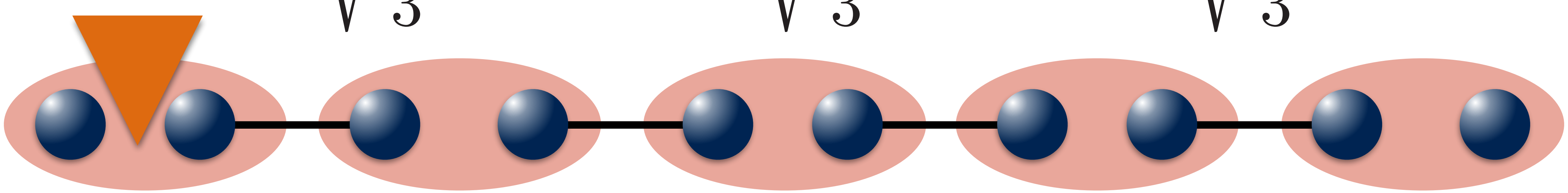

$$\{|\bar{0}\rangle, \cos\theta|\bar{1}\rangle + \sin\theta|\bar{2}\rangle, \sin\theta|\bar{1}\rangle - \cos\theta|\bar{2}\rangle\}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$X \qquad Ze^{-i\theta X} \qquad -Ye^{-i\theta X}$$

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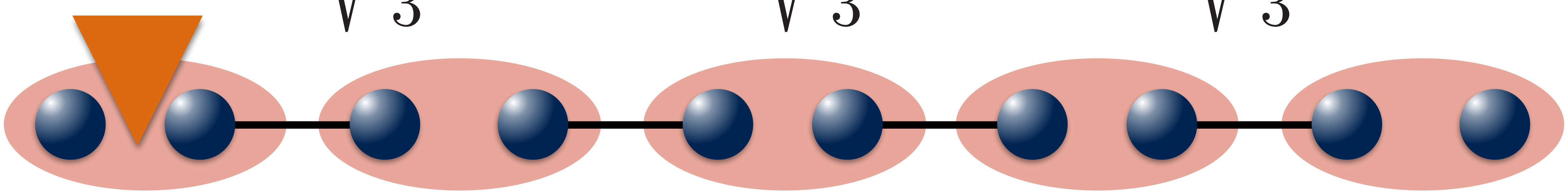

$$\{|\bar{0}\rangle, \cos\theta|\bar{1}\rangle + \sin\theta|\bar{2}\rangle, \sin\theta|\bar{1}\rangle - \cos\theta|\bar{2}\rangle\}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$X \qquad Ze^{-i\theta X} \qquad -Ye^{-i\theta X}$$

MBQC is not deterministic in this case.
← exponentially decaying correlation

More about MBQC on quantum many-body system

- Are measurements necessary?

  → not necessary. We can also employ symmetry breaking field instead.→adiabatic teleportation.

  [Bacon-Flammia, PRL 09; Renes et al, NJP 13]

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- Behavior of two-point correlation function and computational capability → exp. decaying t.p.c. is necessary and sufficient.

  [KF-Morimae, PRA 12]

More about MBQC on quantum many-body system

- Are measurements necessary?

  → not necessary. We can also employ symmetry breaking field instead.→adiabatic teleportation.

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  [KF-Morimae, PRA 12]

- Universal QC → 2D AKLT state with spin-3/2

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- Universal QC at finite temperature → 3D AKLT-like states with spin-5/2, spin-3/2

  [Li et al., PRL '11; KF-Morimae, PRA '12]

Summary

What is quantum order:

→property of ground state

What is symmetry protected topological order

→ground state degeneracy protected by symmetry

How is SPTOs useful for QIP?

→They serve as resources for QIP.

Lecture 3:
topological order in 2D many-body system and quantum error correction codes

Quantum phase transition and information processing

Quantum annealing/ adiabatic quantum computation
[Kadowaki-Nishimori PRE 98; Farhi et al., Science 01]

$$H_{\mathrm{tot}}(s) = (1-s)\underbrace{H_{\mathrm{trivial}}}_{\text{easy to prepare}} + s\underbrace{H_{\mathrm{solution}}}_{\text{hard to find}}$$

(Photo: Clinton Hussy/NASA)
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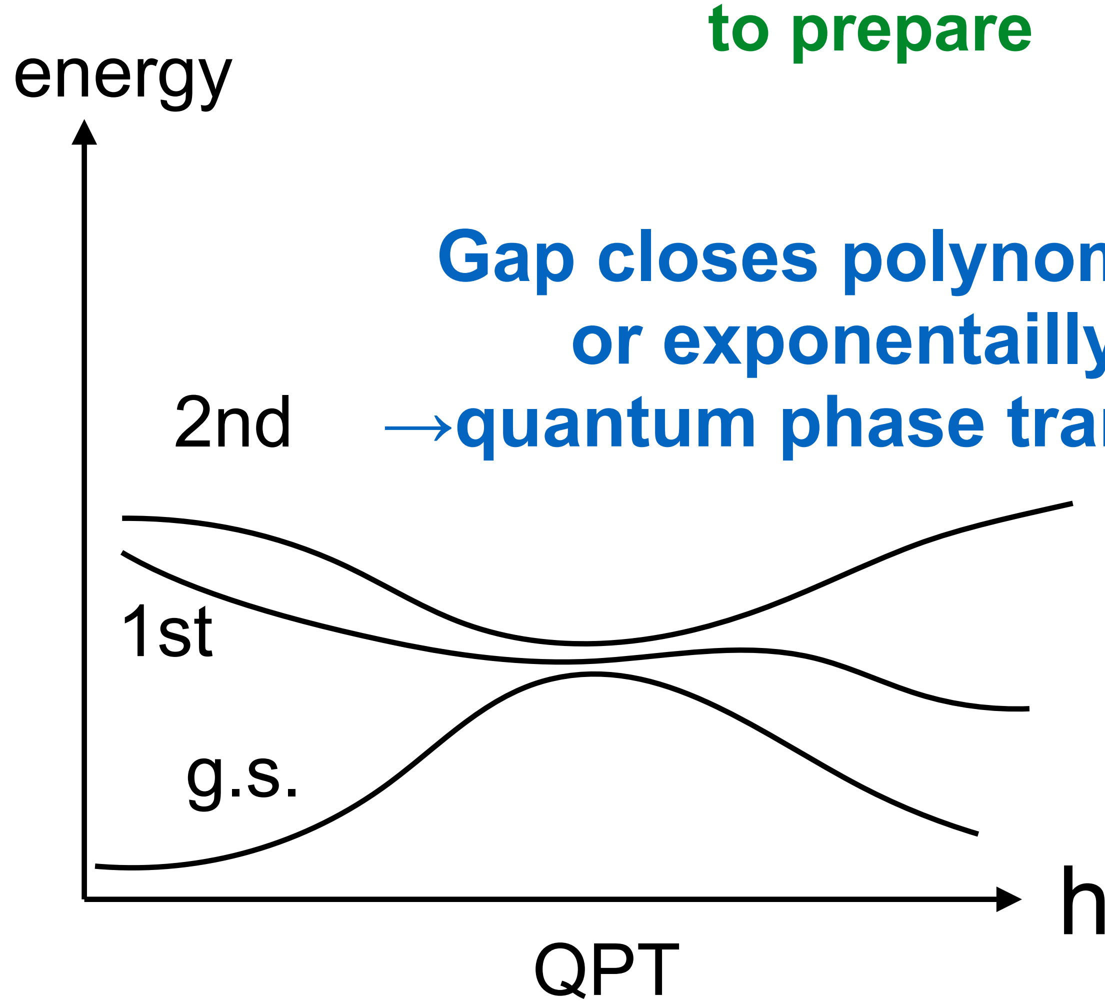

Gap closes polynomially or exponentailly!
→quantum phase transition