Interdisciplinary fields between quantum information science and physics

Keisuke Fujii
The Hakubi center for advanced research/
Graduate School of Science
Kyoto University
Outline of 3 Days

Lecture 1: foundations of quantum computation
- elementary gates and universal quantum computation
- quantum algorithms
- quantum stabilizer formalism (graph state, quantum error correction)

Lecture 2: 1D quantum system
- what is quantum phase
- how it is useful for QIP

Lecture 3: 2D quantum system
- topologically ordered system
- how it is related to quantum error correction codes
- how topologically protected quantum computation works
Application of stabilizer formalism: measurement-based QC
Stabilizer group: $S_n \subset P_n$ Hermitian Abelian subgroup

stabilizer group is specified by the set of generators $\langle \{S_i\} \rangle$.

Stabilizer code states:

$$|\Psi\rangle = S_i|\Psi\rangle \quad \text{for all } S_i \in S_n$$

Logical operators: commute with / independent of the stabilizer group

ex) $S = \langle ZZI, IZZ \rangle$, $L_X = XXXX$, $L_Z = IIIZ$

$\rightarrow \{ |000\rangle, |111\rangle \}$

$IIZ|111\rangle = -|111\rangle$, $XXX|000\rangle = |111\rangle$

logical operators act nontrivially inside the code space
Definition of a graph state

A stabilizer generator is defined for each vertex

\[ K_i = X_i \prod_{j \sim i} Z_j \]

\[ K_i |G\rangle = |G\rangle \text{ for all } i \in V \]

Graph (cluster) state

graph \( G=(V,E) \)

\( V: \text{vertices}, \ E: \text{edges} \)
Definition of a graph state

A stabilizer generator is defined for each vertex

\[ K_i = X_i \prod_{j \sim i} Z_j \]

\[ K_i |G\rangle = |G\rangle \text{ for all } i \in V \]

Graph (cluster) state

\[ |G\rangle = \prod_{e \in E} \Lambda_e(Z) |+\rangle \otimes |V\rangle \]

Graph \( G = (V,E) \)

\( V: \) vertices, \( E: \) edges
Definition of a graph state

A stabilizer generator is defined for each vertex

\[ K_i = X_i \prod_{j \sim i} Z_j \]

\[ K_i |G\rangle = |G\rangle \text{ for all } i \in V \]

CZ gate

\[ |G\rangle = \prod_{e \in E} \Lambda_e(Z) |+\rangle \otimes |V| \]

\[
\left(K_i = \left[ \prod_{e \in E} \Lambda(Z) \right] X_i \left[ \prod_{e \in E} \Lambda(Z) \right] \right)
\]
1D graph (cluster) state

3-qubit 1D graph state

\[
\frac{1}{\sqrt{2}} (|+\rangle|0\rangle|+\rangle + |-\rangle|1\rangle|-\rangle)
\]
1D graph (cluster) state

◆ 3-qubit 1D graph state

\[
\frac{1}{\sqrt{2}} (|+\rangle |0\rangle |+\rangle + |−\rangle |1\rangle |−\rangle )
\]

◆ 4-qubit 1D graph state

\[
\frac{1}{2} (|+\rangle |0\rangle |0\rangle |+\rangle + |+\rangle |0\rangle |1\rangle |−\rangle \\
+ |−\rangle |1\rangle |0\rangle |+\rangle − |−\rangle |1\rangle |1\rangle |−\rangle )
\]
2D cluster state

\[ K_i = X_i Z_{i+n} Z_{i+e} Z_{i+s} Z_{i+w} \]

projective measurement

2D resource state

MBQC
measurement-based quantum computation
2D cluster state

\[ K_i = X_i Z_{i+n} Z_{i+e} Z_{i+s} Z_{i+w} \]

MBQC
measurement-based quantum computation
2D cluster state

\[ K_i = X_i Z_{i+n} Z_{i+e} Z_{i+s} Z_{i+w} \]

MBQC
measurement-based quantum computation

projective measurement

2D resource state
MBQC
measurement-based quantum computation

◆ 2D cluster state

$$K_i = X_i Z_{i+n} Z_{i+e} Z_{i+s} Z_{i+w}$$

projective measurement

2D resource state

space
time
2D cluster state

\[ K_i = X_i Z_{i+n} Z_{i+e} Z_{i+s} Z_{i+w} \]

- Entangling operations are required only offline.
- Provide a connection between many-body physics.
Outline of 3 Days

Lecture 1: foundations of quantum computation
- elementary gates and universal quantum computation
- quantum algorithms
- quantum stabilizer formalism (graph state, quantum error correction)

Lecture 2: 1D quantum system
- what is quantum phase
- how it is useful for QIP

Lecture 3: 2D quantum system
- topologically ordered system
- how it is related to quantum error correction codes
- how topologically protected quantum computation works
Today’s topic

What is quantum phase?

How is it useful for QIP?

How is it realized in a physically natural 1D system?

Keywords: Majorana fermion, symmetry protected topological order, AKLT state
What is quantum phase?

- Quantum phase is a property of ground state (g.s.) of many-body systems.

\[ \langle O \rangle = \langle \Psi_{\text{g.s.}} | O | \Psi_{\text{g.s.}} \rangle \]

Order parameter

\[ H + h \sum_i Z_i \]

Critical point

h
What is quantum phase?

- Quantum phase is a property of ground state (g.s.) of many-body systems.

\[ \langle O \rangle = \langle \Psi_{\text{g.s.}} | O | \Psi_{\text{g.s.}} \rangle \]

order parameter

- The concept of “phase” is robust, and hence it would be useful for quantum information processing.
The degeneracy of g.s. and its robustness against perturbation.

\[ \langle O \rangle = \langle \Psi_{\text{g.s.}} | O | \Psi_{\text{g.s.}} \rangle \]

Order parameter

gapped & degenerated
Degeneracy in g.s.

- The **degeneracy** of g.s. and its robustness against perturbation.

\[
\langle O \rangle = \langle \Psi_{\text{g.s.}} | O | \Psi_{\text{g.s.}} \rangle
\]

**order parameter**

- Quantum information can be encoded into the g.s. and computation can be done inside the g.s.

\[
H + h \sum_i Z_i
\]

• gapped & degenerated

→ how robust? & how computation is done?
1D Ising model

- Ising model with open boundary condition:

\[
H_{\text{Ising}} = - \sum_{i=1}^{N-1} Z_i Z_{i+1}
\]

(recall that \(ZZ|00\rangle = |00\rangle, ZZ|11\rangle = |11\rangle\))

- The g.s. is degenerated: \(\{|0...0\rangle, |1...1\rangle\}\)

\[
S = \prod_i X_i \quad SH_{\text{Ising}} S^\dagger = H_{\text{Ising}}
\]

global spin flip (Z2)
1D Ising model

- Ising model with open boundary condition:

\[
H_{\text{Ising}} = - \sum_{i=1}^{N-1} Z_i Z_{i+1}
\]

(recall that \( ZZ|00\rangle = |00\rangle, ZZ|11\rangle = |11\rangle \))

- The g.s. is degenerated: \( \{ |0...0\rangle, |1...1\rangle \} \)

logical operator \( S = \prod_i X_i \)  

code space

global spin flip (Z2)

\( S H_{\text{Ising}} S^\dagger = H_{\text{Ising}} \)
1D Ising model

• Is the ground state degeneracy robust?

\[ \alpha |0...0\rangle + \beta |1..1\rangle \text{ not robust!} \]
1D Ising model

• Is the ground state degeneracy robust?

\[ \alpha |0\ldots0\rangle + \beta |1\ldots1\rangle \text{ not robust!} \]

\[ H' = H_{\text{Ising}} + \delta \sum_i Z_i \text{ small perturbation} \]
1D Ising model

- Is the ground state degeneracy robust?

\[ \alpha |0...0\rangle + \beta |1..1\rangle \text{ not robust!} \]

\[ H' = H_{\text{Ising}} + \delta \sum_i Z_i \]

Small perturbation

In the large \( N \) limit, the g.s.d. is lifted down, so is not protected against perturbations.
1D Ising model

• Is the ground state degeneracy robust?

\[ \alpha |0...0\rangle + \beta |1..1\rangle \text{ not robust!} \]

\[ H' = H_{\text{Ising}} + \delta \sum_i Z_i \]

small perturbation

In the large \( N \) limit, the g.s.d. is lifted down, so is not protected against perturbations.

Is there a g.s.degeneracy which is robust against perturbations?

Yes.→ topologically ordered system
What is topological order

-a new kind of order in zero-temperature phase of matter.
What is topological order

-a new kind of order in zero-temperature phase of matter.

-cannot be described by Landau’s symmetry breaking argument.
What is topological order

-a new kind of order in zero-temperature phase of matter.

-cannot be described by Landau’s symmetry breaking argument.

-ground states are degenerated and it exhibits long-range quantum entanglement.
What is topological order

-a new kind of order in zero-temperature phase of matter.

-cannot be described by Landau’s symmetry breaking argument.

-ground states are degenerated and it exhibits long-range quantum entanglement.

-the degenerated ground states cannot be distinguished by local operations.
What is topological order

-a new kind of order in zero-temperature phase of matter.

-cannot be described by Landau’s symmetry breaking argument.

-ground states are degenerated and it exhibits long-range quantum entanglement.

-the degenerated ground states cannot be distinguished by local operations.

-topologically ordered states are robust against local perturbations.
What is topological order

-a new kind of order in zero-temperature phase of matter.

cannot be described by Landau’s symmetry breaking argument.

ground states are degenerated and it exhibits long-range quantum entanglement.

-the degenerated ground states cannot be distinguished by local operations.

topologically ordered states are robust against local perturbations.

-related to quantum spin liquids, fractional quantum Hall effect, fault-tolerant quantum computation.
Outline of Lecture 2, 3

Today:

*symmetry* protected topological order in 1D quantum many-body system

Tomorrow:

genuinely topologically ordered system in 2D quantum many-body system and quantum error correction codes
The g.s. degeneracy would be protected by symmetry.
1D Ising model

\[ H' = - \sum_{i=1}^{N-1} Z_i Z_{i+1} + \delta \sum_i Z_i \]

what if this kind of perturbation is prohibited by .... symmetry

The g.s. degeneracy would be protected by symmetry.
Let us consider a mathematically equivalent but physically different system.

Ising model: \[ H' = - \sum_{i=1}^{N-1} Z_i Z_{i+1} \]
Let us consider a mathematically equivalent but physically different system.

Ising model: \[ H' = - \sum_{i=1}^{N-1} Z_i Z_{i+1} \]

Jordan-Wigner transformation (spin ⇔ fermion)

\[ \hat{a}_{2i-1} = X_1 \ldots X_{i-1} Z_i \]
\[ \hat{a}_{2i} = X_1 \ldots X_{i-1} Y_i \]
\[ \{ \hat{a}_k, \hat{a}_{k'} \} = 2 \delta_{k,k'} I \]
(Majorana fermion operator)
1D Majorana fermion

Let us consider a mathematically equivalent but physically different system.

**Ising model**: \( H' = - \sum_{i=1}^{N-1} Z_i Z_{i+1} \)

**Jordan-Wigner transformation** (spin \( \Leftrightarrow \) fermion)

\[
\hat{a}_{2i-1} = X_1 \ldots X_{i-1} Z_i \\
\hat{a}_{2i} = X_1 \ldots X_{i-1} Y_i \\
\{ \hat{a}_k, \hat{a}_{k'} \} = 2 \delta_{k,k'} I
\]
(Majorana fermion operator)

**2N spinless Majorana fermions**: \( H_{\text{Maj}} = - \sum_{j=2}^{N-1} (-i) \hat{a}_{2j} \hat{a}_{2j+1} \)

p-wave superconductor, topological insulator, semiconducting heterostructure (see A. Kitaev and C. Laumann, arXiv:0904.2771 for review)
1D Majorana fermion

\[ H_{\text{Maj}} = - \sum_{j=2}^{N-1} (-i) \hat{a}_{2j} \hat{a}_{2j+1} \]

Ground states: \((-i) \hat{a}_{2i} \hat{a}_{2i+1} |\Psi\rangle = |\Psi\rangle \) for all \(i\).
1D Majorana fermion

\[ H_{\text{Maj}} = - \sum_{j=2}^{N-1} (-i) \hat{a}_{2j} \hat{a}_{2j+1} \]

ground states: \((-i) \hat{a}_{2i} \hat{a}_{2i+1} |\Psi\rangle = |\Psi\rangle\) for all \(i\).

unpaired Majorana fermions \(\circ\) at the edges of the chain.
→ “zero-energy Majorana boundary mode” \(\{ |\bar{0}\rangle, |\bar{1}\rangle \}\)
but fermion operators always appear as a pair!

\[
H_{Maj} = - \sum_{j=2}^{N-1} (-i) \hat{a}_{2j} \hat{a}_{2j+1}
\]

ground states: \((-i) \hat{a}_{2i} \hat{a}_{2i+1} |\Psi\rangle = |\Psi\rangle\) for all \(i\).

unpaired Majorana fermions \(\bullet\) at the edges of the chain.

→ “zero-energy Majorana boundary mode” \(\{|\bar{0}\rangle, |\bar{1}\rangle\}\)

\[
(-i) \hat{a}_1 \hat{a}_{2N} |\bar{0}\rangle = |\bar{1}\rangle, \ (-i) \hat{a}_1 \hat{a}_{2N} |\bar{1}\rangle = |\bar{0}\rangle, \ \hat{a}_1 |\bar{1}\rangle = -|\bar{1}\rangle
\]

\(Y_1 X_2 \ldots X_{2N-1} Y_{2N}\) (Z2 symmetry)

If unpaired Majorana fermions are well separated, this operator would not act.

\(Z_1\)

(act on the ground subspace nontrivially)
1D Majorana fermion

\[ H_{\text{Maj}} = - \sum_{j=2}^{N-1} (-i) \hat{a}_{2j} \hat{a}_{2j+1} \]

ground states: \((-i)\hat{a}_{2i} \hat{a}_{2i+1}\ket{\Psi} = \ket{\Psi}\) for all \(i\).

unpaired Majorana fermions \(\bullet\) at the edges of the chain.

→ “zero-energy Majorana boundary mode” \(\{\ket{\bar{0}}, \ket{\bar{1}}\}\)

\((-i)\hat{a}_1 \hat{a}_{2N}\ket{\bar{0}} = \ket{\bar{1}}, \ (-i)\hat{a}_1 \hat{a}_{2N}\ket{\bar{1}} = \ket{\bar{0}}, \ \hat{a}_1 \ket{\bar{1}} = -\ket{\bar{1}}\)

\(Y_1X_2...X_{2N-1}Y_{2N}\) (Z2 symmetry)

If unpaired Majorana fermions are well separated, this operator would not act.

but fermion operators always appear as a pair!
1D Majorana fermion

Unpaired Majorana fermions (g.s. degeneracy) is robust against any physical perturbation, which preserves the fermionic parity (symmetry).

**Symmetry protected topological (SPT) order**
Majorana fermion

1D p-wave superconductor; spin-orbit-coupled semiconducting wire on s-wave superconductor; topological insulator; cold atom in optical lattice

[Majorana fermion pair creation of Majorana fermions via T-junction]

[Alicea et al., Nature Physics 2011]

[Majorana Fermions in Equilibrium and in Driven Cold-Atom Quantum Wires]

Majorana fermion

1D p-wave superconductor; spin-orbit-coupled semiconducting wire on s-wave superconductor; topological insulator; cold atom in optical lattice

[Non-Abelian statistics and information processing]

1. Majorana fermions

\[ \gamma^1 = \gamma \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij} \]

Features:
- May realize topological (fault tolerant) Clifford gates (non-Abelian anyons)
- Controllable
- Reported to have been realized in lab

2. Time evolution

3. Setup

4. Hadamard gate (braiding)

[Pair creation of Majorana fermions]

[Alicea et al., Nature Physics 2011]
How SPT ordered states are useful for QIP?
1D Cluster Hamiltonian

\[ H_{\text{1dcluster}} = -\sum_{i=2}^{N-1} Z_{i-1} X_i Z_{i+1} \]

stabilizer generators

1D cluster state

\[ X_1 Z_2 \]
\[ Z_1 X_2 Z_3 \]
\[ \ldots \]
\[ Z_{N-2} X_{N-1} Z_N \]
\[ Z_{N-1} X_N \]
$H_{1\text{dcluster}} = - \sum_{i=2}^{N-1} Z_{i-1} X_i Z_{i+1}$

1D cluster state

→The g.s. has 4-fold degeneracy.
$H_{1d \text{cluster}} = - \sum_{i=2}^{N-1} Z_{i-1} X_i Z_{i+1}$
1D Cluster Hamiltonian

\[ H_{1d\text{cluster}} = - \sum_{i=2}^{N-1} Z_{i-1} X_i Z_{i+1} \]

- Global spin flip operators on either odd or even qubits:
  \[ S_o := \prod_k X_{2k-1}, \quad S_e := \prod_k X_{2k} \]

- \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry:
  \[ S_o H_{1d\text{cluster}} S_o^\dagger = H_{1d\text{cluster}}, \quad S_e H_{1d\text{cluster}} S_e^\dagger = H_{1d\text{cluster}} \]
1D Cluster Hamiltonian

\[ H_{1d\text{cluster}} = - \sum_{i=2}^{N-1} Z_{i-1} X_i Z_{i+1} \]

- Global spin flip operators on either odd or even qubits:
  
  \[ S_o := \prod_k X_{2k-1}, \quad S_e := \prod_k X_{2k} \]

- \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry:
  
  \[ S_o H_{1d\text{cluster}} S_o^\dagger = H_{1d\text{cluster}}, \quad S_e H_{1d\text{cluster}} S_e^\dagger = H_{1d\text{cluster}} \]

Any local perturbation commuting with the symmetry operators cannot lift down the g.s. degeneracy.

\[ X_k, \quad Z_k Z_{k+2} \]

[J. Pachos and M. Plenio, PRL 2004; W. Son et al., EPL 2011]
Cluster Phase

\[ H' = H_{1\text{dcluster}} + \delta \sum_i X_i \]

String order parameter

\[ \langle O_{\text{string}} \rangle = \langle \Psi_{\text{gs}} | O_{\text{string}} | \Psi_{\text{gs}} \rangle \]

\[ O_{\text{string}} = \prod_i Z_{i-1}X_iZ_{i+1} \]

\[ = Z_1Y_2X_3...X_{N-2}Y_{N-1}Z_N \]

[J. Pachos and M. Plenio, PRL 2004]
How computation is done in the g.s.?
Matrix product states
Fannes-Nachtergaele-Werner, Comm. Math. ‘92

- Description of quantum states are difficult in general.

\[ |\Psi\rangle = \sum_{i_1,\ldots,i_N} C_{i_1,\ldots,i_N} |i_1\rangle|i_2\rangle\cdots|i_N\rangle \]

exponentially many coefficients!
Matrix product states
Fannes-Nachtergaele-Werner, Comm. Math. ‘92

- Description of quantum states are difficult in general.

\[ |\Psi\rangle = \sum_{i_1, \ldots, i_N} C_{i_1, \ldots, i_N} |i_1\rangle|i_2\rangle \cdots |i_N\rangle \]

exponentially many coefficients!

what if the coefficients have a nice structure?

\[ |\Psi\rangle = \sum_{i_1, \ldots, i_N} \langle R | A[i_n] \cdots A[i_2] A[i_1] |L \rangle \times |i_1\rangle|i_2\rangle \cdots |i_N\rangle \]

the coefficients are denoted by matrix products
Matrix product states
Fannes-Nachtergaele-Werner, Comm. Math. ‘92

- Description of quantum states are difficult in general.

\[ |\Psi\rangle = \sum_{i_1, \ldots, i_N} C_{i_1, \ldots, i_N} |i_1\rangle |i_2\rangle \cdots |i_N\rangle \]

exponentially many coefficients!

what if the coefficients have a nice structure?

\[ |\Psi\rangle = \sum_{i_1, \ldots, i_N} \langle R | A[i_N] \cdots A[i_2] A[i_1] |L\rangle \times |i_1\rangle |i_2\rangle \cdots |i_N\rangle \]

the coefficient are denoted by matrix products

\[ A[0], A[1] \]: matrices to define the coefficient
(in the following, they are 2×2 matrices)
Matrix product states
Fannes-Nachtergaele-Werner, Comm. Math. ‘92

- Description of quantum states are difficult in general.

\[ |\Psi\rangle = \sum_{i_1,\ldots,i_N} C_{i_1,\ldots,i_N} |i_1\rangle|i_2\rangle\cdots|i_N\rangle \]

- exponentially many coefficients!

\[ |\Psi\rangle = \sum_{i_1,\ldots,i_N} \langle R|A[i_n]|\cdots|A[i_2]|A[i_1]|L\rangle \times |i_1\rangle|i_2\rangle\cdots|i_N\rangle \]

- virtual degree (correlation space)
- what if the coefficients have a nice structure?

- physical degree

- the coefficient are denoted by matrix products

\[ A[0], A[1] \] : matrices to define the coefficient (in the following, they are 2×2 matrices)
Examples of MPS

• GHZ state: \( \left( |00\ldots0\rangle + |11\ldots1\rangle \right)/\sqrt{2} \)

\[ \rightarrow A[0] = |0\rangle\langle 0|, \quad A[1] = |1\rangle\langle 1| \]
Examples of MPS

• GHZ state: \( (|00\ldots0\rangle + |11\ldots1\rangle)/\sqrt{2} \)
  \[ \rightarrow A[0] = |0\rangle\langle 0|, \quad A[1] = |1\rangle\langle 1| \]

• W state:
  \( (|100\ldots0\rangle + |010\ldots0\rangle + \cdots |000\ldots1\rangle)/\sqrt{N} \)
  \[ \rightarrow A[0] = I, \quad A[1] = |0\rangle\langle 1| \]
Examples of MPS

- GHZ state: \( (|00\ldots0\rangle + |11\ldots1\rangle) / \sqrt{2} \)
  \[ \rightarrow A[0] = |0\rangle\langle0|, \quad A[1] = |1\rangle\langle1| \]

- W state:
  \[ (|100\ldots0\rangle + |010\ldots0\rangle + \cdots |000\ldots1\rangle) / \sqrt{N} \]
  \[ \rightarrow A[0] = I, \quad A[1] = |0\rangle\langle1| \]

- 1D cluster state: \( A[0] = \sqrt{2}|+\rangle\langle0|, \quad A[1] = \sqrt{2}|−\rangle\langle1| \)
Examples of MPS

- **GHZ state:** \((|00\ldots0\rangle + |11\ldots1\rangle)/\sqrt{2}\)
  
  \[ A[0] = |0\rangle\langle 0|, \quad A[1] = |1\rangle\langle 1| \]

- **W state:**
  \((|100\ldots0\rangle + |010\ldots0\rangle + \cdots |000\ldots1\rangle)/\sqrt{N}\)
  
  \[ A[0] = I, \quad A[1] = |0\rangle\langle 1| \]

- **1D cluster state:** \(A[0] = \sqrt{2}|+\rangle\langle 0|, \quad A[1] = \sqrt{2}|-\rangle\langle 1|\)

\[
\begin{bmatrix}
N-1 \\
\prod_{i=1}^{N-1} \Lambda_{i,i+1}(Z)
\end{bmatrix} |+\rangle^{\otimes N}
\]

\(\Lambda(Z)|11\rangle = -|11\rangle\)

CZ put -1 for neighboring 11

all spin configuration
Examples of MPS

• GHZ state: \( (|00...0\rangle + |11...1\rangle)/\sqrt{2} \)
  \[ A[0] = |0\rangle\langle 0|, \quad A[1] = |1\rangle\langle 1| \]

• W state:
  \( (|100...0\rangle + |010...0\rangle + \cdots |000...1\rangle)/\sqrt{N} \)
  \[ A[0] = I, \quad A[1] = |0\rangle\langle 1| \]

• 1D cluster state:
  \[ A[0] = \sqrt{2}|+\rangle\langle 0|, \quad A[1] = \sqrt{2}|-\rangle\langle 1| \]

\[
\Lambda(Z)|11\rangle = -|11\rangle \\
\text{CZ put -1 for neighboring 11}
\]


\[ \text{all spin configuration} \]

\[ \text{pick up -1} \]
MBQC on 1D cluster model

$$|\Psi_{\text{clus}}\rangle = \sum_{i_1,\ldots,i_N} \langle R | A[i_N] \cdots A[i_2] A[i_1] | L \rangle |i_1 i_2 \ldots i_N\rangle$$

$$A[0] = \sqrt{2} |+\rangle \langle 0|, \quad A[1] = \sqrt{2} |-\rangle \langle 1|$$

Projection on the 1st qubit in the basis

$$\left\{ \frac{1}{\sqrt{2}} (|0\rangle \pm e^{-i\phi} |1\rangle) \right\}$$

edge mode (degeneracy)
MBQC on 1D cluster model

\[ |\Psi_{\text{clus}}\rangle = \sum_{i_1, \ldots, i_N} \langle R| A[i_N] \cdots A[i_2] A[i_1]|L\rangle |i_1 i_2 \ldots i_N\rangle \]

projection on the 1st qubit in the basis

\[ A[0] = \sqrt{2}|+\rangle \langle 0|, \quad A[1] = \sqrt{2}|-\rangle \langle 1| \]

\[ \frac{1}{\sqrt{2}} (|0\rangle \pm e^{-i\phi} |1\rangle) \]

Projection on the 1st qubit in the basis

\[ \frac{1}{\sqrt{2}} (|0\rangle + e^{i\phi} |1\rangle)|\Psi_{\text{clus}}\rangle = H e^{-i\phi Z/2} \]

\[ = \sum_{i_2, \ldots, i_N} \langle R| A[i_N] \cdots A[i_2] (A[0] + e^{i\phi} A[1])/\sqrt{2}|L\rangle |i_2 \ldots i_N\rangle \]

\[ = \sum_{i_2, \ldots, i_N} \langle R| A[i_N] \cdots A[i_2]|L'\rangle |i_2 \ldots i_N\rangle \]

\[ |L'\rangle := H e^{-i\phi Z/2} |L\rangle \]
An arbitrary SU(2) rotation can be implemented.

Byproduct operator due to randomness of the measurement outcomes can be canceled by adoptive measurements.
$H' = H_{1d\text{cluster}} + \delta \sum_i X_i$

String order parameter:

$\langle O_{\text{string}} \rangle = \langle \Psi_{gs} | O_{\text{string}} | \Psi_{gs} \rangle$

[MBQC on 1D cluster model]

[Pachos-Plenio, PRL '04; Doherty-Barrett PRL '09; Fujii et al., PRL '13]
Gate fidelity

\[ F = \frac{(1 + \langle O_{\text{string}}^o \rangle)(1 + \langle O_{\text{string}}^e \rangle)}{4} \]

\[ O_{\text{string}}^{o,e} = \prod_{\text{odd,even}} Z_{i-1}X_iZ_{i+1} \]

\[ H' = H_{1d\text{cluster}} + \delta \sum_i X_i \]

String order parameter

\[ \langle O_{\text{string}} \rangle = \langle \Psi_{gs} | O_{\text{string}} | \Psi_{gs} \rangle \]

Cluster phase

\[ \delta = 1 \]

Critical point

[\text{Pachos-Plenio, PRL '04; Doherty-Barrett PRL '09; Fujii et al., PRL '13}]
Three-Spin Interactions in Optical Lattices and Criticality in Cluster Hamiltonians

Jiannis K. Pachos and Martin B. Plenio

\[ \mathcal{H}_{1d\text{cluster}} = \sum_{i=2}^{N-1} Z_{i-1} X_i Z_{i+1} \]
Can we obtain the resource from natural (two-body) Hamiltonian?
Can we obtain the resource from natural (two-body) Hamiltonian?

→ spin-1 AKLT state
Haldane’s conjecture

- Haldane’s conjecture on 1D Heisenberg anti-ferro model:

\[
H_{\text{Heisenberg}} = \sum_{i} \mathbf{S}_{i} \cdot \mathbf{S}_{i+1}
\]

\[
\mathbf{S}_{i} = (S_{i}^{x}, S_{i}^{y}, S_{i}^{z})
\]

Odd half integer spins, 1/2, 3/2,…
\[\rightarrow \text{massless (gapless)}, \text{critical}\]

Integer spins, 1, 2,…
\[\rightarrow \text{massive (gapped)}, \text{exponentially decaying correlation}\]
Spin-1 1D AKLT (Affleck-Kennedy-Lieb-Tasaki) Hamiltonian

\[ H_{\text{AKLT}} = \sum_i \left[ \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 \right] \]

The g.s. is gapped, exhibits exponentially decaying correlation, and is exactly given as an MPS!
AKLT-model
AKLT, PRL ‘87

- Spin-1 1D AKLT (Affleck-Kennedy-Lieb-Tasaki) Hamiltonian

\[ H_{\text{AKLT}} = \sum_i \left[ \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 \right] \]

The g.s. is gapped, exhibits exponentially decaying correlation, and is exactly given as an MPS!

\[ |\Psi_{\text{AKLT}}\rangle = \sum_{i_1, \ldots, i_N} \langle R|A[i_N]\cdots A[i_2]A[i_1]|L\rangle|i_1\ldots i_N\rangle \quad i_k = 0,1,2 \]
AKLT-model
AKLT, PRL '87

- Spin-1 1D AKLT (Affleck-Kennedy-Lieb-Tasaki) Hamiltonian

\[ H_{\text{AKLT}} = \sum_i \left( \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 \right) \]

The g.s. is gapped, exhibits exponentially decaying correlation, and is exactly given as an MPS!

\[ |\Psi_{\text{AKLT}}\rangle = \sum_{i_1,\ldots,i_N} \langle R|A[i_N] \cdots A[i_2]A[i_1]|L\rangle|i_1\ldots i_N\rangle \]

\[ i_k = 0, 1, 2 \]

2×2 matrices

\[ A[0] = \sqrt{\frac{2}{3}}|0\rangle\langle 1|, \quad A[1] = -\sqrt{\frac{1}{3}}Z, \quad A[2] = -\sqrt{\frac{2}{3}}|1\rangle\langle 0| \]
AKLT-state

2×2 matrices

\[ A[0] = \sqrt{\frac{2}{3}} |0\rangle \langle 1|, \quad A[1] = -\sqrt{\frac{1}{3}} Z, \quad A[2] = -\sqrt{\frac{2}{3}} |1\rangle \langle 0| \]

Projection to spin-1 triplet subspace

\[ \langle 00| \rightarrow \langle 0| \]
\[ \frac{\langle 01| + \langle 10|}{\sqrt{2}} \rightarrow \langle 1| \]

\[ \langle 11| \rightarrow \langle 2| \]

Edge state

Spin-1/2 singlet

\[ \frac{|01\rangle - |10\rangle}{\sqrt{2}} \]
MBQC on AKLT state

\[ |\Psi_{\text{AKLT}}\rangle = \sum_{i_1, \ldots, i_N} \langle R | A[i_N] \cdots A[i_2] A[i_1] | L \rangle |i_1 \ldots i_N\rangle \]

2×2 matrices

\[ A[0] = \sqrt{\frac{2}{3}} |0\rangle \langle 1|, \quad A[1] = -\sqrt{\frac{1}{3}} Z, \quad A[2] = -\sqrt{\frac{2}{3}} |1\rangle \langle 0| \]
\[ |\Psi_{AKLT}\rangle = \sum_{i_1,\ldots,i_N} \langle R | A[i_N] \cdots A[i_2] A[i_1] | L \rangle |i_1\ldots i_N\rangle \]

2×2 matrices
\[
A[0] = \sqrt{\frac{2}{3}} |0\rangle \langle 1|, \quad A[1] = -\sqrt{\frac{1}{3}} Z, \quad A[2] = -\sqrt{\frac{2}{3}} |1\rangle \langle 0|
\]

basis change = local unitary
\[
B[\bar{0}] = \sqrt{\frac{1}{3}} X, \quad B[\bar{1}] = \sqrt{\frac{1}{3}} Z, \quad B[\bar{2}] = \sqrt{\frac{1}{3}} Y
\]
MBQC on AKLT state

\[ |\Psi_{AKLT}\rangle = \sum_{i_1, \ldots, i_N} \langle R| A[i_N] \cdots A[i_2] A[i_1]|L\rangle |i_1 \ldots i_N\rangle \]

2\times2 matrices

\[ A[0] = \sqrt{\frac{2}{3}}|0\rangle\langle 1|, \quad A[1] = -\sqrt{\frac{1}{3}}Z, \quad A[2] = -\sqrt{\frac{2}{3}}|1\rangle\langle 0| \]

basis change = local unitary

\[ B[\bar{0}] = \sqrt{\frac{1}{3}}X, \quad B[\bar{1}] = \sqrt{\frac{1}{3}}Z, \quad B[\bar{2}] = \sqrt{\frac{1}{3}}Y \]

\{ |\bar{0}\rangle, \cos \theta |\bar{1}\rangle + \sin \theta |\bar{2}\rangle, \sin \theta |\bar{1}\rangle - \cos \theta |\bar{2}\rangle \}
MBQC on AKLT state

\[ |\Psi_{\text{AKLT}}\rangle = \sum_{i_1, \ldots, i_N} \langle R | A[i_N] \cdots A[i_2] A[i_1] | L \rangle |i_1 \ldots i_N\rangle \]

\[ 2\times2 \text{ matrices} \quad A[0] = \sqrt{\frac{2}{3}} |0\rangle \langle 1|, \quad A[1] = -\sqrt{\frac{1}{3}} Z, \quad A[2] = -\sqrt{\frac{2}{3}} |1\rangle \langle 0| \]

basis change = local unitary

\[ B[\bar{0}] = \sqrt{\frac{1}{3}} X, \quad B[\bar{1}] = \sqrt{\frac{1}{3}} Z, \quad B[\bar{2}] = \sqrt{\frac{1}{3}} Y \]

\[ \{|\bar{0}\rangle, \cos \theta |\bar{1}\rangle + \sin \theta |\bar{2}\rangle, \sin \theta |\bar{1}\rangle - \cos \theta |\bar{2}\rangle \} \]

\[ X \quad Z e^{-i\theta X} \quad -Y e^{-i\theta X} \]

MBQC is not deterministic in this case. ← exponentially decaying correlation
More about MBQC on quantum many-body system

- Are measurements necessary?
  → not necessary. We can also employ symmetry breaking field instead. → adiabatic teleportation.
  [Bacon-Flammia, PRL 09; Renes et al, NJP 13]
More about MBQC on quantum many-body system

• Are measurements necessary?
  → not necessary. We can also employ symmetry breaking field instead. → adiabatic teleportation.
  [Bacon-Flammia, PRL 09; Renes et al, NJP 13]

• Behavior of two-point correlation function and computational capability → exp. decaying t.p.c. is necessary and sufficient.
  [KF-Morimae, PRA 12]
More about MBQC on quantum many-body system

• Are measurements necessary?
  → not necessary. We can also employ symmetry breaking field instead. → adiabatic teleportation.
  [Bacon-Flammia, PRL 09; Renes et al, NJP 13]

• Behavior of two-point correlation function and computational capability
  → exp. decaying t.p.c. is necessary and sufficient.
  [KF-Morimae, PRA 12]

• Universal QC → 2D AKLT state with spin-3/2
  [Chen et al., PRL ’09; Miyake, Ann. Phys. ’11; Wei et al., PRL ’11]
More about MBQC on quantum many-body system

• Are measurements necessary?
  → not necessary. We can also employ symmetry breaking field instead. → adiabatic teleportation.
  [Bacon-Flammia, PRL 09; Renes et al, NJP 13]

• Behavior of two-point correlation function and computational capability
  → exp. decaying t.p.c. is necessary and sufficient.
  [KF-Morimae, PRA 12]

• Universal QC → 2D AKLT state with spin-3/2
  [Chen et al., PRL ’09; Miyake, Ann. Phys. ’11; Wei et al., PRL ’11]

• Universal QC at finite temperature → 3D AKLT-like states with spin-5/2, spin-3/2
  [Li et al., PRL ’11; KF-Morimae, PRA ’12]
What is quantum order:
→property of ground state

What is symmetry protected topological order
→ground state degeneracy protected by symmetry

How is SPTOs useful for QIP?
→They serve as resources for QIP.

Lecture 3:
topological order in 2D many-body system and quantum error correction codes
Quantum phase transition and information processing

Quantum annealing/adiabatic quantum computation
[Kadowaki-Nishimori PRE 98; Farhi et al., Science 01]

\[ H_{\text{tot}}(s) = (1 - s)H_{\text{trivial}} + sH_{\text{solution}} \]

easy to prepare

hard to find

(Photo: Clinton Hussy/NASA)
Quantum phase transition and information processing

Quantum annealing/adiabatic quantum computation
[Kadowaki-Nishimori PRE 98; Farhi et al., Science 01]

$$H_{\text{tot}}(s) = (1 - s)H_{\text{trivial}} + sH_{\text{solution}}$$

easy to prepare  hard to find

Gap closes polynomially or exponentially!

$\rightarrow$ quantum phase transition

QPT

energy

1st

g.s.

2nd

(Photo: Clinton Hussy/NASA)